LESSON 3-1 SOME BEGINNING MATH FACTS

OVERVIEW

This lesson reviews some prerequisite mathematics terms and skills you should already know including rules for positive integer exponents.

REAL NUMBERS AND THEIR SUBSETS

A **real** number is any number that can be located on a ruler like number line. The set of real numbers includes

- Whole numbers (0, 1, 2, 3, …)
- Integers (… −3, −2, −1, 0, 1, 2, 3, …)
- Rational numbers (fractions with an integer numerator and a nonzero integer denominator)
- Irrational numbers (numbers that are not rational such as π, $\sqrt{3}$, and any nonrepeating, nonending decimal number such as 0.1010010001 …)

CONSTANTS AND VARIABLES

A **constant** is a quantity that always has a fixed value like the number of inches in one foot. A **variable** is a quantity that can change in value such as the number of hours of sleep a person gets from week to week. Variables in algebra are commonly represented by letters such as x and y. A number that multiplies a variable is called its **coefficient**. In the formula $F = \frac{9}{5}C + 32$, which tells how to convert from degrees Celsius to degrees Fahrenheit,

- C is a variable
- $\frac{9}{5}$ is the coefficient of C

- 32 is a constant

COMPARISON SYMBOLS

Table 3.1 summarizes the symbols used to compare two quantities.

Table 3.1. Comparison Symbols

Symbol	Translation	Example
$=$	is equal to	$5 = 5$
$\neq$	is *not* equal to	$5 \neq 3$
$>$	is greater than	$5 > 3$
$\geq$	is greater than *or* equal to	$x \geq 5$ means that x can be 5 *or* any number greater than 5.
$<$	is less than	$3 < 5$
$\leq$	is less than *or* equal to	$x \leq 3$ means that x can be 3 *or* any number less than 3.

Since 7 is between 6 and 8, you can write $6 < 7 < 8$. The inequality $6 < 7 < 8$ means that 6 is less than 7 and, at the same time, 7 is less than 8. In general, if x is between a and b with $a < b$, then $a < x < b$. The inequality $a \leq x \leq b$ means that x is between a and b, or may be equal to either a or b, as shown in Figure 3.1.

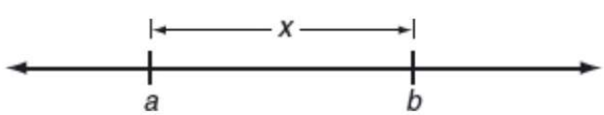

Figure 3.1 $a \leq x \leq b$

PRODUCTS AND FACTORS

When two or more quantities are multiplied together, the answer is called the **product**. Each of the quantities being multiplied together is called a **factor** of that product. Since $2 \times 3 \times 4 = 24$, the numbers 2, 3, and 4 are each factors of the 24.

EXPONENT RULES

The expression x^n is read as "x raised to the nth power" where the *exponent* n tells how many times the *base* x is used in the repeated multiplication of x:

$$\underbrace{x \cdot x \cdot x \cdots x \cdot x}_{x \text{ appears as a factor } n \text{ times}} = x^n$$

where n is a positive integer and $x^1 = x$. From this definition, there are several useful exponent rules that follow when the bases are the same.

RULES FOR POSITIVE INTEGER EXPONENTS (n and m are positive integers and x and y are real numbers)		
Product Rule	**Quotient Rule**	**Power Rule**
$x^n \cdot x^m = x^{n+m}$	$x^n + x^m = x^{n-m}$	$(x^n)^m = x^{n \times m}$ and $(xy)^m = x^m \cdot y^m$

Here are some examples:

- $(2p^4)^3 = 2^3 \cdot (p^4)^3 = 8p^{12}$
- $\left(\frac{x}{2}\right)^4 = \frac{x^4}{2^4} = \frac{x^4}{16}$
- $\frac{r^{m+1}}{r^2} = r^{m-1}$
- $\frac{x^7 y^2}{x^3 y} = x^{7-3} \cdot y^{2-1} = x^4 y$

➡ Example

If $3x - 2y = 5$, what is the value of $\frac{8^x}{4^y}$?

Solution

Change the fraction into an equivalent fraction having the same base in both the numerator and the denominator:

$$\begin{aligned}\frac{8^x}{4^y} &= \frac{(2^3)^x}{(2^2)^y}\\ &= \frac{2^{3x}}{2^{2y}}\\ &= 2^{3x-2y} \leftarrow \text{Substitute 5 for } 3x-2y\\ &= 2^5\\ &= 32\end{aligned}$$

➡ Example

If $3^{k+1} - 3^k = m$, what is 3^{k+2} in terms of m?

(A) $\frac{4}{3}m$

(B) $3m$

(C) $\frac{9}{2}$

(D) $9m$

Solution

- Rewrite the left side of the given equation in terms of 3^k:

$$\begin{aligned}3\cdot 3^k - 3^k &= m\\ 2\cdot 3^k &= m\\ 3^k &= \frac{m}{2}\end{aligned}$$

- Multipy 3^k by 3^2 to get 3^{k+2} :

$$\begin{aligned}3^2(3^k) &= 3^2\left(\frac{m}{2}\right)\\ 3^{k+2} &= \frac{9m}{2}\end{aligned}$$

The correct choice is **(C)**.

DISTRIBUTIVE LAW

Two times the sum of x and y can be represented by $2(x + y)$, which must be equivalent to multiplying x and y individually by 2 and then adding the products together, as in $2x + 2y$. Thus, $2(x + y) = 2x + 2y$.

MATH REFERENCE FACT

When applying the distributive law, remove parentheses by multiplying each term inside the parentheses by the term outside the parentheses:

$$a(x + y) = ax + ay$$

Here are some more examples:

- $3(a - 2b) = 3a - 6b$
- $x(xy + 4) = x^2y + 4x$
- $rs^2\left(\frac{1}{r} + \frac{1}{s}\right) = \frac{rs^2}{r} + \frac{rs^2}{s} = s^2 + rs$

The process of reversing the distributive law is called **factoring**:

- $4x + 4y = 4(x + y)$ ← Factor out 4
- $2x + 3x = x(2 + 3)$ ← Factor out x
  $= 5x$
- $2 - x = -(x - 2)$ ← Factor out –1
- $a^2b - ab^2 = ab(a - b)$ ← Factor out ab

ROOTS

The square root notation $\sqrt{x}$ means one of two equal nonnegative numbers whose product is x as in $\sqrt{16} = 4$ since $4 \times 4 = 16$.

- $\sqrt[2]{x}$ means the same thing as $\sqrt{x}$.

- $\sqrt[3]{x}$ means the cube root of x, which is one of three equal numbers whose product is x as in $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. Also, $\sqrt[4]{x}$ represents the fourth root of x, $\sqrt[5]{x}$ is the fifth root of x, and so on.
- The square root symbol $\sqrt{\ }$ is called a **radical** and always represents the *positive* or **principal square root** of the number underneath it. Thus, $\sqrt{9} = 3$ but $\sqrt{9} \neq -3$ despite the fact that $(-3) \times (-3) = 9$. However, if $x^2 = 9$, then either $x = -3$ or $x = 3$.

Some Properties of Square Root Radicals

- To multiply square root radicals together, multiply the radicands:

$$\sqrt{3} \times \sqrt{7} = \sqrt{21}$$

and

$$\sqrt{2}\left(\sqrt{8} + \sqrt{5}\right) = \sqrt{16} + \sqrt{10} = 4 + \sqrt{10}$$

- To simplify a square root radical, factor the radicand as the product of two numbers, one of which is the greatest perfect square factor of the radicand. Then write the radical over each factor and simplify:

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

- To combine square root radicals, the number or expression underneath the radical sign, called the **radicand**, must be the same:

$$\begin{aligned} \sqrt{18} + 5\sqrt{2} &= \sqrt{9 \cdot 2} + 5\sqrt{2} \\ &= 3\sqrt{2} + 5\sqrt{2} \\ &= 8\sqrt{2} \end{aligned}$$

MATH REFERENCE FACT

$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$. Radicals cannot be distributed over the operations of addition and subtraction. For example:

$$\sqrt{4 + 9} \neq \sqrt{4} + \sqrt{9}$$

■ To eliminate a radical from the denominator of a fraction, multiply the numerator and denominator of the fraction by the radical:

$$\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

ALGEBRAIC REPRESENTATION

The SAT may include problems that test your ability to use a combination of variables, constants, and arithmetic operations to represent a given situation.

➡ Example

Sam and Jeremy have ages that are consecutive odd integers. The product of their ages is 783. Which equation could be used to find Jeremy's age, j, if he is the younger man?

(A) $j^2 + 2 = 783$
(B) $j^2 - 2 = 783$
(C) $j^2 + 2j = 783$
(D) $j^2 - 2j = 783$

Solution

Since consecutive odd integers differ by 2, if Jeremy's age is represented by j, then $j + 2$ represents Sam's age so $j(j + 2) = 783$. Using the distributive law, $j^2 + 2j = 783$.

The correct choice is **(C)**.

➡ Example

A crew of painters are hired to paint a house. The cost of the paint is estimated to be \$250 and each of the p painters will get paid \$15 per hour. If the crew of painters will work a 7 hour day and the job is estimated to take d days, which expression best represents the total cost of painting the house?

(A) $250 + \frac{15p}{7d}$

(B) $250 + 105pd$

(C) $250 + \frac{105}{pd}$

(D) $7d(250 + 15p)$

Solution

- The hourly cost of the p painters for each 7 hour day is $(p \times 15) \times 7 = 105p$.
- Since the job is estimated to take d days, the cost of the painter is $105pd$.
- Adding the cost of the paint gives $250 + 105pd$.

The correct choice is **(B)**.

LESSON 3-1 TUNE-UP EXERCISES

Multiple-Choice

1. If $5 = a^x$, then $\frac{5}{a} =$

   (A) a^{x+1}
   (B) a^{x-1}
   (C) a^{1-x}
   (D) $a^{\frac{x}{5}}$

2. If $y = 25 - x^2$ and $1 \le x \le 5$, what is the smallest possible value of y?

   (A) 0
   (B) 1
   (C) 5
   (D) 10

3. Given $y = wx^2$ and y is not 0. If the values of x and w are each doubled, then the value of y is multiplied by

   (A) 1
   (B) 2
   (C) 4
   (D) 8

4. If $\frac{x^{23}}{x^m} = x^{15}$ and $(x^4)^n = x^{20}$, then $mn =$

   (A) 13
   (B) 24
   (C) 28
   (D) 40

5. If $2 = p^3$, then $8p$ must equal

   (A) p^6
   (B) p^8
   (C) p^{10}
   (D) $8\sqrt{2}$