THE FOUR MATHEMATICS CONTENT AREAS

ANSWER SHEET FOR TUNE-UP EXERCISES

For Reference only.

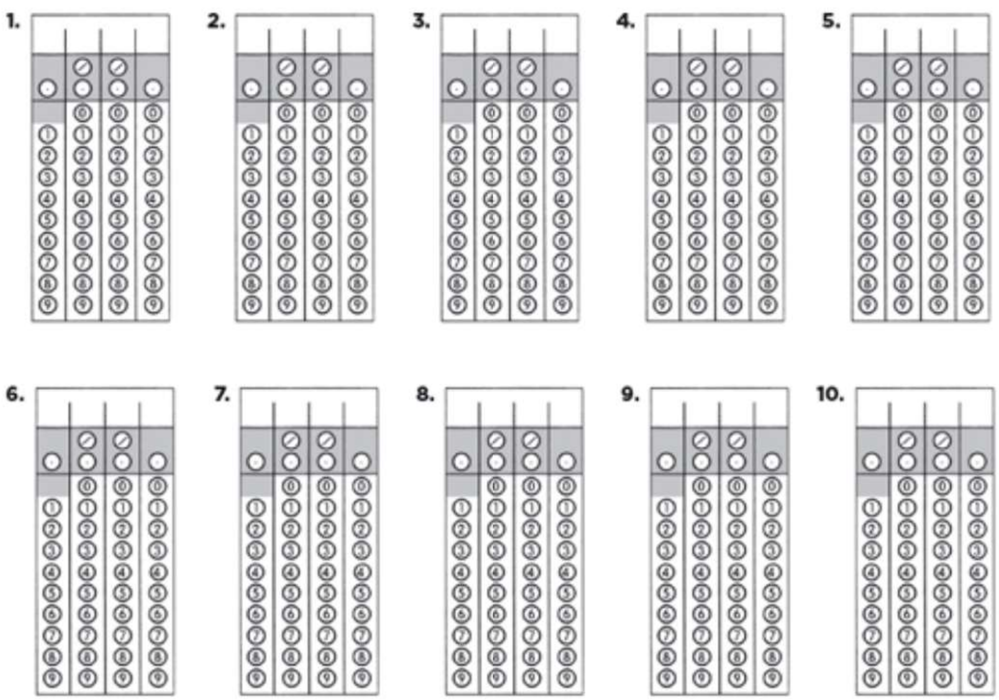
Lesson # ______

Before you work through the SAT Tune-Up exercises found at the end of each review lesson, you may wish to reproduce the answer form below. This may contain more circles and grids than you will need for a particular exercise section. Mark only those you need and leave the rest blank.

Multiple-Choice

1. Ⓐ Ⓑ Ⓒ Ⓓ	6. Ⓐ Ⓑ Ⓒ Ⓓ	11. Ⓐ Ⓑ Ⓒ Ⓓ	16. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ	7. Ⓐ Ⓑ Ⓒ Ⓓ	12. Ⓐ Ⓑ Ⓒ Ⓓ	17. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ	8. Ⓐ Ⓑ Ⓒ Ⓓ	13. Ⓐ Ⓑ Ⓒ Ⓓ	18. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ	9. Ⓐ Ⓑ Ⓒ Ⓓ	14. Ⓐ Ⓑ Ⓒ Ⓓ	19. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ	10. Ⓐ Ⓑ Ⓒ Ⓓ	15. Ⓐ Ⓑ Ⓒ Ⓓ	20. Ⓐ Ⓑ Ⓒ Ⓓ

Grid-In



3

Heart of Algebra

"Heart of Algebra" represents the first of the four major mathematics content groups tested by the redesigned SAT. The new SAT places particular emphasis on mastery of the core algebraic concepts and techniques covered in this chapter with more than one-third of the SAT Math questions based on this content area.

LESSONS IN THIS CHAPTER

Lesson 3-1 Some Beginning Math Facts
Lesson 3-2 Solving Linear Equations
Lesson 3-3 Equations with More Than One Variable
Lesson 3-4 Polynomials and Algebraic Fractions
Lesson 3-5 Factoring
Lesson 3-6 Quadratic Equations
Lesson 3-7 Systems of Equations
Lesson 3-8 Algebraic Inequalities
Lesson 3-9 Absolute Value Equations and Inequalities
Lesson 3-10 Graphing in the xy-Plane
Lesson 3-11 Graphing Linear Systems
Lesson 3-12 Working with Functions

LESSON 3-1 SOME BEGINNING MATH FACTS

OVERVIEW

This lesson reviews some prerequisite mathematics terms and skills you should already know including rules for positive integer exponents.

REAL NUMBERS AND THEIR SUBSETS

A **real** number is any number that can be located on a ruler like number line. The set of real numbers includes

- Whole numbers (0, 1, 2, 3, …)
- Integers (… −3, −2, −1, 0, 1, 2, 3, …)
- Rational numbers (fractions with an integer numerator and a nonzero integer denominator)
- Irrational numbers (numbers that are not rational such as π, $\sqrt{3}$, and any nonrepeating, nonending decimal number such as 0.1010010001 …)

CONSTANTS AND VARIABLES

A **constant** is a quantity that always has a fixed value like the number of inches in one foot. A **variable** is a quantity that can change in value such as the number of hours of sleep a person gets from week to week. Variables in algebra are commonly represented by letters such as x and y. A number that multiplies a variable is called its **coefficient**. In the formula $F = \frac{9}{5}C + 32$, which tells how to convert from degrees Celsius to degrees Fahrenheit,

- C is a variable
- $\frac{9}{5}$ is the coefficient of C

- 32 is a constant

COMPARISON SYMBOLS

Table 3.1 summarizes the symbols used to compare two quantities.

Table 3.1. Comparison Symbols

Symbol	Translation	Example
$=$	is equal to	$5 = 5$
$\neq$	is *not* equal to	$5 \neq 3$
$>$	is greater than	$5 > 3$
$\geq$	is greater than *or* equal to	$x \geq 5$ means that x can be 5 *or* any number greater than 5.
$<$	is less than	$3 < 5$
$\leq$	is less than *or* equal to	$x \leq 3$ means that x can be 3 *or* any number less than 3.

Since 7 is between 6 and 8, you can write $6 < 7 < 8$. The inequality $6 < 7 < 8$ means that 6 is less than 7 and, at the same time, 7 is less than 8. In general, if x is between a and b with $a < b$, then $a < x < b$. The inequality $a \leq x \leq b$ means that x is between a and b, or may be equal to either a or b, as shown in Figure 3.1.

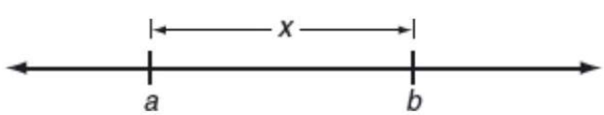


Figure 3.1 $a \leq x \leq b$

PRODUCTS AND FACTORS

When two or more quantities are multiplied together, the answer is called the **product**. Each of the quantities being multiplied together is called a **factor** of that product. Since $2 \times 3 \times 4 = 24$, the numbers 2, 3, and 4 are each factors of the 24.

EXPONENT RULES

The expression x^n is read as "x raised to the nth power" where the *exponent* n tells how many times the *base* x is used in the repeated multiplication of x:

$$\underbrace{x \cdot x \cdot x \cdots x \cdot x}_{x \text{ appears as a factor } n \text{ times}} = x^n$$

where n is a positive integer and $x^1 = x$. From this definition, there are several useful exponent rules that follow when the bases are the same.

RULES FOR POSITIVE INTEGER EXPONENTS (*n* and *m* are positive integers and *x* and *y* are real numbers)		
Product Rule	**Quotient Rule**	**Power Rule**
$x^n \cdot x^m = x^{n+m}$	$x^n \div x^m = x^{n-m}$	$(x^n)^m = x^{n \times m}$ and $(xy)^m = x^m \cdot y^m$

Here are some examples:

- $(2p^4)^3 = 2^3 \cdot (p^4)^3 = 8p^{12}$
- $\left(\frac{x}{2}\right)^4 = \frac{x^4}{2^4} = \frac{x^4}{16}$
- $\frac{r^{m+1}}{r^2} = r^{m-1}$
- $\frac{x^7y^2}{x^3y} = x^{7-3} \cdot y^{2-1} = x^4y$

➡ Example

If $3x - 2y = 5$, what is the value of $\frac{8^x}{4^y}$?

Solution

Change the fraction into an equivalent fraction having the same base in both the numerator and the denominator:

$$\frac{8^x}{4^y} = \frac{(2^3)^x}{(2^2)^y}$$
$$= \frac{2^{3x}}{2^{2y}}$$
$$= 2^{3x-2y} \leftarrow \text{Substitute 5 for } 3x-2y$$
$$= 2^5$$
$$= 32$$

➡ Example

If $3^{k+1} - 3^k = m$, what is 3^{k+2} in terms of m?

(A) $\frac{4}{3}m$

(B) $3m$

(C) $\frac{9}{2}$

(D) $9m$

Solution

- Rewrite the left side of the given equation in terms of 3^k:

$$3 \cdot 3^k - 3^k = m$$
$$2 \cdot 3^k = m$$
$$3^k = \frac{m}{2}$$

- Multipy 3^k by 3^2 to get 3^{k+2} :

$$3^2(3^k) = 3^2\left(\frac{m}{2}\right)$$
$$3^{k+2} = \frac{9m}{2}$$

The correct choice is **(C)**.

DISTRIBUTIVE LAW

Two times the sum of x and y can be represented by $2(x + y)$, which must be equivalent to multiplying x and y individually by 2 and then adding the products together, as in $2x + 2y$. Thus, $2(x + y) = 2x + 2y$.

> **MATH REFERENCE FACT**
>
> When applying the distributive law, remove parentheses by multiplying each term inside the parentheses by the term outside the parentheses:
>
> $$a(x + y) = ax + ay$$

Here are some more examples:

- $3(a - 2b) = 3a - 6b$
- $x(xy + 4) = x^2y + 4x$
- $rs^2\left(\frac{1}{r} + \frac{1}{s}\right) = \frac{rs^2}{r} + \frac{rs^2}{s} = s^2 + rs$

The process of reversing the distributive law is called **factoring**:

- $4x + 4y = 4(x + y)$ ← Factor out 4
- $2x + 3x = x(2 + 3)$ ← Factor out x
  $= 5x$
- $2 - x = -(x - 2)$ ← Factor out –1
- $a^2b - ab^2 = ab(a - b)$ ← Factor out ab

ROOTS

The square root notation $\sqrt{x}$ means one of two equal nonnegative numbers whose product is x as in $\sqrt{16} = 4$ since $4 \times 4 = 16$.

- $\sqrt[2]{x}$ means the same thing as $\sqrt{x}$.

- $\sqrt[3]{x}$ means the cube root of x, which is one of three equal numbers whose product is x as in $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. Also, $\sqrt[4]{x}$ represents the fourth root of x, $\sqrt[5]{x}$ is the fifth root of x, and so on.
- The square root symbol $\sqrt{\ }$ is called a **radical** and always represents the *positive* or **principal square root** of the number underneath it. Thus, $\sqrt{9} = 3$ but $\sqrt{9} \neq -3$ despite the fact that $(-3) \times (-3) = 9$. However, if $x^2 = 9$, then either $x = -3$ or $x = 3$.

Some Properties of Square Root Radicals

- To multiply square root radicals together, multiply the radicands:

$$\sqrt{3} \times \sqrt{7} = \sqrt{21}$$

and

$$\sqrt{2}\left(\sqrt{8} + \sqrt{5}\right) = \sqrt{16} + \sqrt{10} = 4 + \sqrt{10}$$

- To simplify a square root radical, factor the radicand as the product of two numbers, one of which is the greatest perfect square factor of the radicand. Then write the radical over each factor and simplify:

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

- To combine square root radicals, the number or expression underneath the radical sign, called the **radicand**, must be the same:

$$\begin{aligned}\sqrt{18} + 5\sqrt{2} &= \sqrt{9 \cdot 2} + 5\sqrt{2} \\ &= 3\sqrt{2} + 5\sqrt{2} \\ &= 8\sqrt{2}\end{aligned}$$

MATH REFERENCE FACT

$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$. Radicals cannot be distributed over the operations of addition and subtraction. For example:

$$\sqrt{4 + 9} \neq \sqrt{4} + \sqrt{9}$$

- To eliminate a radical from the denominator of a fraction, multiply the numerator and denominator of the fraction by the radical:

$$\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

ALGEBRAIC REPRESENTATION

The SAT may include problems that test your ability to use a combination of variables, constants, and arithmetic operations to represent a given situation.

➡ Example

Sam and Jeremy have ages that are consecutive odd integers. The product of their ages is 783. Which equation could be used to find Jeremy's age, j, if he is the younger man?

(A) $j^2 + 2 = 783$
(B) $j^2 - 2 = 783$
(C) $j^2 + 2j = 783$
(D) $j^2 - 2j = 783$

Solution

Since consecutive odd integers differ by 2, if Jeremy's age is represented by j, then $j + 2$ represents Sam's age so $j(j + 2) = 783$. Using the distributive law, $j^2 + 2j = 783$.

The correct choice is **(C)**.

➡ Example

A crew of painters are hired to paint a house. The cost of the paint is estimated to be $250 and each of the p painters will get paid $15 per hour. If the crew of painters will work a 7 hour day and the job is estimated to take d days, which expression best represents the total cost of painting the house?

(A) $250 + \frac{15p}{7d}$

(B) $250 + 105pd$

(C) $250 + \frac{105}{pd}$

(D) $7d(250 + 15p)$

Solution

- The hourly cost of the p painters for each 7 hour day is $(p \times 15) \times 7 = 105p$.
- Since the job is estimated to take d days, the cost of the painter is $105pd$.
- Adding the cost of the paint gives $250 + 105pd$.

The correct choice is **(B)**.

LESSON 3-1 TUNE-UP EXERCISES

Multiple-Choice

1. If $5 = a^x$, then $\frac{5}{a} =$

   (A) a^{x+1}
   (B) a^{x-1}
   (C) a^{1-x}
   (D) $a^{\frac{x}{5}}$

2. If $y = 25 - x^2$ and $1 \le x \le 5$, what is the smallest possible value of y?

   (A) 0
   (B) 1
   (C) 5
   (D) 10

3. Given $y = wx^2$ and y is not 0. If the values of x and w are each doubled, then the value of y is multiplied by

   (A) 1
   (B) 2
   (C) 4
   (D) 8

4. If $\frac{x^{23}}{x^m} = x^{15}$ and $(x^4)^n = x^{20}$, then $mn =$

   (A) 13
   (B) 24
   (C) 28
   (D) 40

5. If $2 = p^3$, then $8p$ must equal

   (A) p^6
   (B) p^8
   (C) p^{10}
   (D) $8\sqrt{2}$

6. If $10^{k-3} = m$, then $10^k =$

(A) $1{,}000m$
(B) $m + 1{,}000$
(C) $\frac{m}{1{,}000}$
(D) $m - 1{,}000$

7. If w is a positive number and $w^2 = 2$, then $w^3 =$

(A) $\sqrt{2}$
(B) $2\sqrt{2}$
(C) 4
(D) $3\sqrt{2}$

8. If $x = \sqrt{6}$ and $y^2 = 12$, then $\frac{4}{xy} =$

(A) $\frac{3}{2\sqrt{2}}$
(B) $\frac{\sqrt{2}}{3}$
(C) $\frac{3}{\sqrt{2}}$
(D) $\frac{2\sqrt{2}}{3}$

9. If x is a positive integer such than $x^9 = r$ and $x^5 = w$, which of the following must be equal to x^{13}?

(A) $rw - 1$
(B) $r + w - 1$
(C) $\frac{r^2}{w}$
(D) $r^2 - w$

10. Caitlin has a movie rental card worth \$175. After she rents the first movie, the card's value is \$172.25. After she rents the second movie, its value is \$169.50. After she rents the third movie, the card is worth \$166.75. Assuming the pattern continues, which of the following equations define A, the amount of money on the rental card after n rentals?

(A) $175 - 2.75n$
(B) $2.75n - 175$
(C) $(175 - 2.75)n$
(D) $\frac{175}{2.75}n$

11. Three times the sum of a number and four is equal to five times the number, decreased by two. If x represents the number, which equation is a correct translation of the statement?

(A) $3(x + 4) = 5x - 2$
(B) $3(x + 4) = 5(x - 2)$
(C) $3x + 4 = 5x - 2$
(D) $3x + 4 = 5(x - 2)$

12. Owen gets paid \$280 per week plus 5% commission on all sales for selling electronic equipment. If he sells d dollars worth of electronic equipment in one week, which algebraic expression represents the amount of money he will earn in w weeks?

(A) $(280d + 5)w$
(B) $280 + 0.05dw$
(C) $(280 + 0.05d)w$
(D) $280w + 0.05d$

13. Which expression represents the number of hours in w weeks and d days?

(A) $7w + 12d$
(B) $84w + 24d$
(C) $168w + 24d$
(D) $168w + 60d$

14. Which verbal expression can be represented by $2(x - 5)$?

(A) 5 less than 2 times x
(B) 2 multiplied by x less 5
(C) twice the difference of x and 5
(D) the product of 2 and x, decreased by 5

15. If k pencils cost c cents, what is the cost in cents of p pencils?

(A) $\frac{pc}{k}$

(B) $\frac{pk}{c}$

(C) $\frac{c}{kp}$

(D) cpk

16. If it takes h hours to paint a rectangular wall that is x feet wide and y feet long, how many *minutes* does it take to paint 1 square foot of the bulletin board?

(A) $\frac{xy}{60h}$

(B) $\frac{60h}{xy}$

(C) $\frac{h}{60}(x+y)$

(D) $\frac{hxy}{60}$

17. If Carol earns x dollars a week for 3 weeks and y dollars a week for 4 weeks, what is the average number of dollars per week that she earns?

(A) $\frac{1}{7}(x+y)$

(B) $\frac{3x+4y}{7}$

(C) $\frac{12xy}{7}$

(D) $\frac{x}{3}+\frac{y}{4}$

18. Tim bought a skateboard and two helmets for a total of d dollars. If the skateboard costs s dollars, the cost of each helmet could be represented by which of the following expressions?

(A) $2ds$

(B) $\frac{ds}{2}$

(C) $\frac{d-s}{2}$

(D) $d-\frac{s}{2}$

19. The length of a rectangle is three feet less than twice its width. If x represents the width of the rectangle, in feet, which inequality represents the area of the rectangle that is *at most* 30 square feet?

(A) $x(2x - 3) \leq 30$
(B) $x(2x - 3) \geq 30$
(C) $x(3 - 2x) \leq 30$
(D) $x(3 - 2x) \geq 30$

20. In 1995, the U.S. federal government paid off one-third of its debt. If the original amount of the debt was $4,920,000,000,000, which expression represents the amount that was *not* paid off?

(A) 1.64×10^4
(B) 1.64×10^{12}
(C) 3.28×10^8
(D) 3.28×10^{12}

$$\frac{\left(b^{2n+1}\right)^3}{b^n \cdot b^{5n+1}}$$

21. The expression above is equivalent to which of the following?

(A) b^2
(B) $\frac{1}{b^2}$
(C) b^{2n}
(D) $\frac{1}{b^{2n}}$

Grid-In

1. If $2^4 \times 4^2 = 16^x$, then $x =$

2. If $a^7 = 7{,}777$ and $\frac{a^6}{b} = 11$, what is the value of ab?

3. If $y = 2^{2p-1}$ and $z = p - 2$, what is the value of $\frac{y}{z}$ when $p = 2.5$?

4. If $13 \leq k \leq 21$, $9 \leq p \leq 19$, $2 < m < 6$, and k, p, and m are integers, what is the largest possible value of $\frac{k-p}{m}$?

LESSON 3-2 SOLVING LINEAR EQUATIONS

OVERVIEW

A **linear equation** is an equation in which each term is either a constant or the product of a constant and the first power of a single variable such as $2x + 1 = 7$. To solve a linear equation, isolate the variable on one side of the equation by doing the same thing on both sides until the resulting equation has the form, *variable* = constant, as in $x = 3$.

BASIC TYPES OF SAT EQUATIONS

Some basic types of equations that appear on the SAT may look a little different than the equations encountered in your regular mathematics classes, but they can be solved using the same procedures.

➡ Example

If $\frac{r}{s} = 6$, what is the value of $\frac{4s}{r}$?

Solution

If $\frac{r}{s} = 6$, then inverting both sides gives $\frac{s}{r} = \frac{1}{6}$:

$$\frac{4s}{r} = 4\left(\frac{s}{r}\right) = 4\left(\frac{1}{6}\right) = \frac{4}{6} = \frac{2}{3}$$

Grid-in **2/3**

TIP

In your regular math class, solving for *x* typically gives you the final answer to the problem. But *x* is not always the final answer on the SAT. When reading a question, circle or underline the quantity that the question asks you to find. After you solve the problem, check that your answer matches what you were required to find by looking back at what you circled or underlined.

➡ Example

If $kx - 19 = k - 1$ and $k = 3$, what is the value of $x + k$?

Solution

Replace k with 3 and solve the resulting equation in the usual way:

$$3x - 19 = 3 - 1$$
$$3x - 19 = 2$$
$$3x = 21$$
$$\frac{3x}{3} = \frac{21}{3}$$
$$x = 7$$
$$x + k = 7 + 3 = 10$$

Grid-in **10**

➡ Example

If $2y - 7 = 18$, what is the value of $2y + 3$?

(A) 15
(B) 18
(C) 27
(D) 28

Solution

Rather than first solving the equation for y, solve for the quantity asked for directly. To get $2y + 3$ from $2y - 7$, add 10 to both sides of the equation:

$$2(y - 7) + 10 = 18 + 10$$
$$2y + 3 = 28$$

The correct choice is **(D)**.

➡ Example

When 6 times a number x is added to 5, the result is 19. What number results when 5 is subtracted from 3 times x?

(A) -3

(B) $\frac{2}{3}$

(C) 2
(D) 7

Solution

According to the conditions of the problem, $6x + 5 = 19$, and you must find the value of $3x - 5$.

- If $6x + 5 = 19$, then $6x = 19 - 5$ so $6x = 14$.
- Dividing each side of $6x = 14$ by 2 gives $3x = 7$, and subtracting 5 from each side yields $3x - 5 = 2$.

The correct choice is **(C)**.

➡ Example

If $\frac{5}{x} = \frac{9}{x+12}$, what is the value of $\frac{x}{3}$?

Solution

If If $\frac{5}{x} = \frac{9}{x+12}$, then cross-multiplying makes $5(x + 12) = 9x$. Remove the parentheses by multiplying each term inside the parentheses by 5:

$$5x + 60 = 9x$$

$$4x = 60 \leftarrow \text{To get } \frac{x}{3} \text{ from } 4x, \text{ divide both sides by } 12$$

$$\frac{4x}{12} = \frac{60}{12}$$

$$\frac{x}{3} = 5$$

Grid-in **5**

➥ **Example**

If If $\frac{13}{16}x - \frac{3}{8}x = \frac{2}{5} + \frac{3}{10}$, what is the value of x?

(A) $\frac{17}{8}$

(B) $\frac{9}{4}$

(C) $\frac{8}{5}$

(D) $\frac{17}{10}$

Solution

Combine fractions on each side of the equation:

$$\frac{13}{16}x - \frac{3}{8}x = \frac{2}{5} + \frac{3}{10}$$

$$\frac{13}{16}x - \frac{6}{16}x = \frac{4}{10} + \frac{3}{10}$$

$$\frac{7}{16}x = \frac{7}{10} \quad \leftarrow \text{Isolate } x \text{ by multiplying both sides by the reciprocal of its coefficient}$$

$$\frac{16}{7}\left(\frac{7}{16}x\right) = \frac{16}{\cancel{7}}\left(\frac{\overset{1}{\cancel{7}}}{10}\right)$$

$$1 \cdot x = \frac{16}{10}$$

$$x = \frac{8}{5}$$

The correct choice is **(C)**.

Solving a Linear Equation Using Two Operations

To isolate a variable in a linear equation, it may be necessary to add or subtract and then to multiply or divide. If an equation contains parentheses, remove them by multiplying each term inside the parentheses by the term outside the parentheses.

➥ Example

Solve for *b*:

$$3(b + 2) + 2b = 21$$

Solution

$$3(b + 2) + 2b = 21$$

- Remove the parentheses by multiplying each term inside the parentheses by 3: $3b + 6 + 2b = 21$
- Combine like terms: $5b + 6 = 21$
- Subtract 6 from each side of the equation: $5b + 6 - 6 = 21 - 6$
- Simplify: $5b = 15$
- Divide each side of the equation by 5: $\frac{5b}{5} = \frac{15}{5}$

$$b = 3$$

➥ Example

$$\frac{7}{3}\left(x + \frac{9}{28}\right) - 3 = 17$$

Which value of *x* satisfies the equation above?

(A) $\frac{33}{4}$

(B) $\frac{249}{28}$

(C) $\frac{77}{4}$

(D) $\frac{45}{28}$

Solution

$$\frac{7}{3}\left(x+\frac{9}{28}\right)=20$$

$$\frac{7}{3}x+\left(\frac{\cancel{7}}{\cancel{3}}\right)\left(\frac{\cancel{9}^{3}}{\cancel{28}_{4}}\right)=20$$

$$\frac{7}{3}x=20-\frac{3}{4}$$

$$\frac{7}{3}x=\frac{80}{4}-\frac{3}{4}$$

$$\frac{3}{7}\cdot\left(\frac{7}{3}x\right)=\frac{3}{\cancel{7}}\cdot\left(\frac{\cancel{77}^{11}}{4}\right)$$

$$x=\frac{33}{4}$$

The correct choice is **(A)**.

➡ Example

$$\frac{2(m+4)+13}{5}=\frac{21-(8-3m)}{4}$$

In the equation above, what is the value of *m*?

Solution

- Simplify the numerators:

$$\frac{2m+21}{5}=\frac{13+3m}{4}$$

- Cross-multiply and simplify:

$$5(13+3m)=4(2m+21)$$

$$65+15m=8m+84$$

- Isolate the variable:

$$15m-8m=84-65$$

$$7m=19$$

$$m=\frac{19}{7}$$

Grid-in **19/7**

LESSON 3-2 TUNE-UP EXERCISES

Multiple-Choice

1. If $\frac{m-n}{n} = \frac{4}{9}$, what is the value of $\frac{n}{m}$?

   (A) $\frac{9}{13}$

   (B) $\frac{7}{4}$

   (C) $\frac{9}{5}$

   (D) $\frac{13}{7}$

2. If $\frac{p+4}{p-4}$, what is the value of p?

   (A) $\frac{2}{3}$

   (B) $\frac{10}{13}$

   (C) $\frac{28}{9}$

   (D) $\frac{14}{3}$

3. If $4x + 7 = 12$, what is the value of $8x + 3$?

   (A) 9
   (B) 11
   (C) 13
   (D) 15

4. $\frac{6}{x} = \frac{4}{x-9}$, what is the value of $\frac{x}{18}$?

   (A) 3
   (B) 2
   (C) $\frac{1}{2}$
   (D) $\frac{3}{2}$

5. If $3j - (k + 5) = 16 - 4k$, what is the value of $j + k$?

(A) 8
(B) 7
(C) 5
(D) 4

6. If $\frac{1}{2}(10p + 2) = p + 7$, then $4p =$

(A) 6
(B) $\frac{5}{2}$
(C) 4
(D) 3

7. If $0.25y + 0.36 = 0.33y - 1.48$, what is the value of $\frac{y}{10}$?

(A) 2.30
(B) 1.40
(D) 0.75
(C) 0.64

8. If $\frac{4}{7}k = 36$, then $\frac{3}{7}k =$

(A) 21
(B) 27
(C) 32
(D) 35

9. If $\frac{1}{2}x + \frac{1}{4}x + \frac{1}{8}x = 14$, then $x =$

(A) 4
(B) 8
(C) 12
(D) 16

10. If $\frac{2}{x} = 2$, then $x + 2 =$

(A) $\frac{3}{2}$
(B) $\frac{5}{2}$
(C) 3
(D) 4

11. If $\frac{y-2}{2} = y + 2$, then $y =$

(A) −6
(B) −4
(C) −2
(D) 4

12. If $\frac{2y}{7} = \frac{y+3}{4}$, then $y =$

(A) 5
(B) 9
(C) 13
(D) 21

13. If $\frac{y}{3} = 4$, then $3y =$

(A) 4
(B) 12
(C) 24
(D) 36

14. When the number k is multiplied by 5, the result is the same as when 5 is added to k. What is the value of k?

(A) $\frac{4}{5}$
(B) 1
(C) $\frac{5}{4}$
(D) $\frac{3}{2}$

$$\frac{8r+7}{4s} = 11$$

15. If $\frac{1}{2}r + 1 = s + 1$, what is the value of $r + s$ for the equation above?

(A) $\frac{1}{2}$
(B) $\frac{3}{4}$
(C) 1
(D) $\frac{3}{2}$

16. If $m + 1 = \frac{5(m-1)}{3}$, then $\frac{1}{m}$

(A) $\frac{1}{4}$

(B) $\frac{3}{8}$

(C) 2

(D) $\frac{8}{3}$

$$\frac{5(p-1)+6}{8} = \frac{7-(3-2p)}{12}$$

17. In the equation above, what is the value of p?

(A) $\frac{1}{3}$

(B) $\frac{5}{11}$

(C) $\frac{2}{3}$

(D) $\frac{9}{11}$

18. During the investigation of an archeological dig, a femur bone was found and used to estimate the height of the person it came from using the formula $h = 61.4 + 2.3F$, where h is the height, in *centimeters*, of a person whose femur is F *centimeters* in length. Using this formula, the height of the person was estimated to be 5 feet 8 inches. The length of the femur, in centimeters, was closest to which of the following lengths? [Note: 1 inch = 2.54 centimeters]

(A) 37.3
(B) 48.4
(C) 51.0
(D) 56.3

19. Last month, Sara, Ryan, and Taylor received a total of 882 emails. If Sara received 25% more emails than the sum of the number of emails received by Ryan and Taylor, how many emails did Sara receive?

(A) 448
(B) 486
(C) 490
(D) 504

20. In an election for senior class president, Emily received approximately 25% more votes than Alexis. If Emily received 163 votes, the number of votes Alexis received is closest to

(A) 122
(B) 130
(C) 138
(D) 204

21. At City High School, the sophomore class has 60 more students than the freshman class. The junior class has 50 fewer students than twice the students in the freshman class. The senior class is three times as large as the freshman class. If there are a total of 1,424 students at City High School, how many students are in the freshman class?

(A) 202
(B) 205
(C) 235
(D) 236

Grid-In

1. If $2w - 1 = 2$, what is the value of $w^2 - 1$?

2. If $11 - 3x$ is 4 less than 13, what is the value of $6x$?

3. If $2x + 1 = 8$ and $15 - 3y = 0$, what is value of $\frac{x}{y}$?

4. The total score in a football game was 72 points. The winning team scored 12 points more than the losing team. How many points did the winning team score?

5. Max’s cell phone plan charges a monthly “pay-as-you-go rate” of \$0.13 for each text message sent or received. If Max was charged \$7.41 for 15 more sent texts than he received, how many texts did Max send?

6. If $\frac{7}{12}x - \frac{1}{3}x = \frac{1}{2} + \frac{3}{8}$, what is the value of x?

$$\frac{5(y-2)}{y} - \frac{1}{3} = 0$$

7. In the equation above, what is the value of y?

$$D = 141 - 0.16p$$

$$S = 64 + 0.28p$$

8. The set of equations above describes how the supply, S, and demand, D, for a computer memory chip depends on market price. If p represents the price in dollars for each lot of 10 memory chips, for what dollar price per memory chip does supply equal demand?

9. If $\frac{7(x+9)}{4} - 1 = 41$, what is the value of $x - 9$?

10. Gerald and Jim work at a furniture store. Gerald is paid \$185 per week plus 4% of his total sales. Jim is paid \$275 per week plus 2.5% of his total sales. What amount of sales will make their weekly pay the same?

11. If 7 quarters and n nickels is equivalent to 380 pennies, what is the value of n?

12. A gardener is planting two types of trees:

    - Type A is three feet tall and grows at a rate of 15 inches per year.
    - Type B is five feet tall and grows at a rate of 9 inches per year.

    How many years will it take for these trees to grow to the same height?

$$\frac{9n-(5n-3)}{8} = \frac{2(n-1)-(3-7n)}{12}$$

13. In the equation above, what is the value of n?

14. An animal shelter spends \$2.35 per day to care for each cat and \$5.50 per day to care for each dog. If \$89.50 is spent caring for a total of 22 cats and dogs, how much was spent on caring for all of the cats?

$$r = 2.24 + 0.06x$$

$$p = 2.89 + 0.10x$$

15. In the equations above, r and p represent the price per gallon, in dollars, of regular and premium grades of gasoline, respectively, x months after January 1 of last year. What was the cost per gallon, in dollars, of premium gasoline for the month in which the per gallon price of premium exceeded the per gallon price of regular by \$0.93?

LESSON 3-3 EQUATIONS WITH MORE THAN ONE VARIABLE

OVERVIEW

An equation that has more than one letter can have many different numerical solutions. For example, in the equation $x + y = 8$, if $x = 1$, then $y = 7$; if $x = 2$, then $y = 6$. Since the value of y depends on the value of x, and x may be any real number, the equation $x + y = 8$ has infinitely many solutions.

The SAT may include questions that ask you to

- Find the value of an expression that is a multiple of one side of an equation. For example, if $x + 2y = 9$, then the value of $2x + 4y$ is 18 since

$$2x + 4y = 2(x + 2y) = 2(9) = 18$$

- Solve for one letter in terms of the other letter(s) of an equation. For example, if $x + y = 8$, then x in terms of y is $x = 8 - y$.

WORKING WITH AN EQUATION IN TWO UNKNOWNS

Some SAT questions can be answered by multiplying or dividing an equation by a suitable number.

➥ Example

If $3x + 3y = 12$, what is the value of $x + y - 6$?

Solution

- Solve the given equation for $x + y$ by dividing each member by 3:

$$\frac{3x}{3} + \frac{3y}{3} = \frac{12}{3}$$
$$x + y = 4$$

- Evaluate the required expression by replacing $x + y$ with 4:

$$x + y - 6 = 4 - 6$$
$$= -2$$

The value of $x + y - 6$ is -2.

➡ Example

If $\frac{t}{s} = \frac{3}{4}$, what is the value of $\frac{3s}{4t}$?

Solution

Since $\frac{t}{s} = \frac{3}{4}$, then $3s = 4t$ $s \cdot \frac{3s}{4t} = 1$.

➡ Example

If $x + 5 = t$, then $2x + 9 =$

(A) $t - 1$
(B) $t + 1$
(C) $2t$
(D) $2t - 1$

Solution

METHOD 1: Use algebraic reasoning:

- Since the coefficient of x in $2x + 9$ is 2, multiply the given equation by 2:

$$2(x + 5) = 2t, \text{ so } 2x + 10 = 2t$$

- Comparing the left side of $2x + 10 = 2t$ with $2x + 9$ suggests that you subtract 1 from both sides of the equation:

$$(2x + 10) - 1 = 2t - 1$$
$$2x + 9 = 2t - 1$$

The correct choice is **(D)**.

METHOD 2: Pick easy numbers for x and t that satisfy $x + 5 = t$. Then use these values for x and t to compare $2x + 9$ with each of the answer choices. For example, when $x = 1$, $1 + 5 = t = 6$. Using $x = 1$, you know that $2x + 9 = 2(1) + 9 = 11$. Hence, the correct answer choice is the one that evaluates to 11 when $t = 6$:

Choice (A):	$t - 1 = 6 - 1$	$= 5$	✗
Choice (B):	$t + 1 = 6 + 1$	$= 7$	✗
Choice (C):	$2t = 2(6)$	$= 12$	✗
Choice (D):	$2t - 1 = 2(6) - 1$	$= 11$	✓

➡ Example

If $\frac{4}{x+1} = \frac{8}{y-2}$ where $x \neq -1$ and $y \neq 2$, what is y in terms of x ?

(A) $y = 2x + 1$
(B) $y = 2(x + 2)$
(C) $y = 2x$
(D) $y = 4x + 2$

Solution

$$\frac{4}{x+1} = \frac{8}{y-2}$$
$$4(y - 2) = 8(x + 1)$$
$$4y - 8 = 8x + 8$$
$$4y = 8x + 16$$
$$\frac{4y}{4} = \frac{8x}{4} + \frac{16}{4}$$
$$y = 2x + 4$$
$$y = 2(x + 2)$$

The correct choice is **(B)**.

➥ Example

$$3(r + 2s) = r + s$$

If (r, s) is a solution to the equation above and $s \neq 0$, what is the value of $\frac{r}{s}$?

(A) $-\frac{5}{2}$

(B) $-\frac{2}{5}$

(C) $\frac{2}{3}$

(D) $\frac{3}{2}$

Solution

Removing the parentheses gives $3r + 6s = r + s$. Collecting like terms on the same side of the equation makes $2r = -5s$ so $r = -\frac{5}{2}s$ and $\frac{r}{s} = -\frac{5}{2}$.

The correct choice is **(A)**.

LESSON 3-3 TUNE-UP EXERCISES

Multiple-Choice

1. If $V = \frac{1}{3}Bh$, what is h expressed in terms of B and V?

   (A) $\frac{1}{3}VB$

   (B) $\frac{V}{3B}$

   (C) $\frac{3V}{B}$

   (D) $3VB$

2. If $F = \frac{kmM}{r^2}$, then $m =$

   (A) $\frac{Fr^2}{kM}$

   (B) $\frac{kFr^2}{M}$

   (C) $\frac{kM}{Fr^2}$

   (D) $F(r^2 + kM)$

3. If $P = 2(L + W)$, what is W in terms of P and L?

   (A) $P - 2L$

   (B) $\frac{P - 2L}{2}$

   (C) $\frac{2L - P}{2}$

   (D) $\frac{1}{2}(P - L)$

4. If $A = \frac{1}{2}h(x + y)$, what is y in terms of A, h, and x?

   (A) $\frac{2A - hx}{h}$

   (B) $\frac{A - hx}{2h}$

   (C) $2Ah - x$

   (D) $2A - hx$

5. If $s = \frac{2x+t}{r}$, then $x =$

(A) $\frac{rs-t}{2}$

(B) $\frac{rs+1}{2}$

(C) $2rs - t$

(D) $rs - 2t$

6. If $x = x_0 + \frac{1}{2}(v + v_0)t$, what is v in terms of the other variables?

(A) $\frac{2(x - x_0)}{v_0 t}$

(B) $\frac{2(x - x_0)}{t} - v_0$

(C) $\frac{t(x - x_0)}{2v_0}$

(D) $v_0 t - \frac{2(x - x_0)}{t}$

7. If $2s - 3t = 3t - s$, what is s in terms of t?

(A) $\frac{t}{2}$

(B) $2t$

(C) $t + 2$

(D) $\frac{t}{2} + 1$

8. If $xy + z = y$, what is x in terms of y and z?

(A) $\frac{y+z}{y}$

(B) $\frac{y-z}{z}$

(C) $\frac{y-z}{y}$

(D) $1 - z$

9. If $b(x + 2y) = 60$ and $by = 15$, what is the value of bx?

(A) 15
(B) 20
(C) 25
(D) 30

10. If $\frac{a-b}{b}=\frac{2}{3}$, what is the value of $\frac{a}{b}$?

(A) $\frac{1}{2}$

(B) $\frac{3}{5}$

(C) $\frac{3}{2}$

(D) $\frac{5}{3}$

11. If $s + 3s$ is 2 more than $t + 3t$, then $s - t =$

(A) -2

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) $\frac{3}{4}$

12. If $\frac{1}{p+q} = r$ and $p \neq -q$, what is p in terms of r and q?

(A) $\frac{rq-1}{q}$

(B) $\frac{1+rq}{q}$

(C) $\frac{r}{1+rq}$

(D) $\frac{1-rq}{r}$

13. If $\frac{a+b+c}{3}=\frac{a+b}{2}$ then $c =$

(A) $\frac{a-b}{2}$

(B) $\frac{a+b}{2}$

(C) $5a+5b$

(D) $\frac{a+b}{5}$

14. If the value of n nickels plus d dimes is c cents, what is n in terms of d?

(A) $\frac{c}{5} - 2d$

(B) $5c - 2d$

(C) $\frac{c-d}{10}$

(D) $\frac{cd}{10}$

15. If $\frac{c}{d} - \frac{a}{b} = x$, $a = 2c$, and $b = 5d$, what is the value of $\frac{c}{d}$ in terms of x?

(A) $\frac{2}{3}x$

(B) $\frac{3}{4}x$

(C) $\frac{4}{3}x$

(D) $\frac{5}{3}x$

$$\frac{4}{t-1} = \frac{2}{w-1}$$

16. If in the equation above $t \neq 1$ and $w \neq 1$, then $t =$

(A) $2w - 1$
(B) $2(w - 1)$
(C) $w - 2$
(D) $2w$

Grid-In

1. If $16 \times a^2 \times 64 = (4 \times b)^2$ and a and b are positive integers, then b is how many times greater than a?

2. If $3a - c = 5b$ and $3a + 3b - c = 40$, what is the value of b?

3. If $a = 2x + 3$ and $b = 4x - 7$, for what value of x is $3b = 5a$?

$$\frac{x}{8} + \frac{y}{5} = \frac{31}{40}$$

4. In the equation above, if x and y are positive integers, what is the value of $x + y$?

5. If $\frac{1}{8}x + \frac{1}{8}y = y - 2x$, then what is the value of $\frac{x}{y}$?

LESSON 3-4 POLYNOMIALS AND ALGEBRAIC FRACTIONS

OVERVIEW

A **polynomial** is a single term or the sum or difference of two or more unlike terms. For example, the polynomial $a + 2b + 3c$ represents the sum of the three unlike terms a, $2b$, and $3c$. Since polynomials represent real numbers, they can be added, subtracted, multiplied, and divided using the laws of arithmetic.

Whenever a letter appears in the denominator of a fraction, you may assume that it cannot represent a number that makes the denominator of the fraction equal to 0.

CLASSIFYING POLYNOMIALS

A polynomial can be classified according to the number of terms it contains.

- A polynomial with one term, as in $3x^2$, is called a **monomial**.
- A polynomial with two unlike terms, as in $2x + 3y$, is called a **binomial**.
- A polynomial with three unlike terms, as in $x^2 + 3x - 5$, is called a **trinomial**.

OPERATIONS WITH POLYNOMIALS

Polynomials may be added, subtracted, multiplied, and divided.

- To add polynomials, write one polynomial on top of the other one so that like terms are aligned in the same vertical columns. Then combine like terms. For example:

$$\begin{array}{r} 2x^2 - 3x + 7 \\ +\ \ x^2 + 5x - 9 \\ \hline 3x^2 + 2x - 2 \end{array}$$

■ To subtract polynomials, take the opposite of each term of the polynomial that is being subtracted. Then add the two polynomials. For instance, the difference

$$(7x - 3y - 9z) - (5x + y - 4z)$$

can be changed into a sum by adding the opposite of each term of the second polynomial to the first polynomial:

$$\begin{array}{rr} & 7x - 3y - 9z \\ + & -5x - y + 4z \\ \hline & 2x - 4y - 5z \end{array}$$

■ To multiply monomials, multiply their numerical coefficients and multiply *like* variable factors by *adding* their exponents. For example:

$$\begin{aligned}(-2a^2b)(4a^3b^2) &= (-2)(4)(a^2 \cdot a^3)(b \cdot b^2) \\ &= -8(a^{2+3})(b^{1+2}) \\ &= -8a^5b^3\end{aligned}$$

■ To divide monomials, divide their numerical coefficients and divide *like* variable factors by *subtracting* their exponents. For example:

$$\begin{aligned}\frac{14x^5y^2}{21x^2y^2} &= \left(\frac{14}{21}\right)\left(\frac{x^5}{x^2}\right)\left(\frac{y^2}{y^2}\right) \\ &= \left(\frac{2}{3}\right)(x^{5-2})(1) \\ &= \frac{2}{3}x^3\end{aligned}$$

■ To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the resulting products. For example:

$$\begin{aligned}5x(3x - y + 2) &= 5x(3x) + 5x(-y) + 5x(2) \\ &= 15x^2 - 5xy + 10x\end{aligned}$$

The expression $-(a - b)$ can be interpreted as “take the opposite of whatever is inside the parentheses.” The result is $b - a$ since

$$\begin{aligned}-(a-b) &= -1(a-b)\\ &= (-1)a + (-1)(-b)\\ &= -a + b \text{ or } b - a\end{aligned}$$

- To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and add the resulting quotients. For example:

$$\frac{6x+15}{3} = \frac{6x}{3} + \frac{15}{3} = 2x + 5$$

MULTIPLYING BINOMIALS USING FOIL

To find the product of two binomials, write them next to each other and then add the products of their *F*irst, *O*uter, *I*nner, and *L*ast pairs of terms.

➥ Example

Multiply $(2x + 1)$ by $(x - 5)$.

Solution

For the product $(2x + 1)(x - 5)$, $2x$ and x are the *F*irst pair of terms; $2x$ and -5 are the *O*utermost pairs of terms; 1 and x are the *I*nnermost terms; 1 and -5 are the *L*ast terms of the binomials. Thus:

$$\begin{aligned}(2x+1)(x-5) &= \overset{\text{F}}{(2x)(x)} + \overset{\text{O}}{(2x)(-5)} + \overset{\text{I}}{(1)(x)} + \overset{\text{L}}{(1)(-5)}\\ &= 2x^2 \quad + [-10x + x] \quad - 5\\ &= 2x^2 \quad - \quad 9x \quad - 5\end{aligned}$$

PRODUCTS OF SPECIAL PAIRS OF BINOMIALS

SAT problems may involve these special products: $(a - b)(a + b)$, $(a + b)^2$, and $(a - b)^2$.

➥ Example

Express $(x + 3)(x - 3)$ as a binomial.

TIP

Time Saver

You can save some time if you memorize and learn to recognize when these special multiplication rules can be applied.

- $\boldsymbol{(a - b)(a + b) = a^2 - b^2}$
- $\boldsymbol{(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2}$
- $\boldsymbol{(a - b)^2 = (a - b)(a - b) = a^2 - 2ab - b^2}$

Solution

$$\begin{aligned}(x + 3)(x - 3) &= (x)^2 - (3)^2 \\ &= x^2 - 9\end{aligned}$$

➥ Example

Express $(2y - 1)(2y + 1)$ as a binomial.

Solution

$$\begin{aligned}(2y - 1)(2y + 1) &= (2y)^2 - (1)^2 \\ &= 4y^2 - 1\end{aligned}$$

➥ Example

If $(x + y)^2 - (x - y)^2 = 28$, what is the value of xy?

Solution

- Square each binomial:

$$\begin{aligned}(x + y)^2 - (x - y)^2 &= 28 \\ (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2) &= 28\end{aligned}$$

- Write the first squared binomial without the parentheses. Then remove the second set of parentheses by changing the sign of each term inside the parentheses to its opposite.

$$x^2 + 2xy + y^2 - x^2 + 2xy - y^2 = 28$$

- Combine like terms. Adding x^2 and $-x^2$ gives 0, as does adding y^2 and $-y^2$. The sum of $2xy$ and $2xy$ is $4xy$. The result is

$$4xy = 28$$
$$xy = \frac{28}{4} = 7$$

COMBINING ALGEBRAIC FRACTIONS

Algebraic fractions are combined in much the same way as fractions in arithmetic.

➡ Example

Write $\frac{w}{2} - \frac{w}{3}$ as a single fraction.

Solution

The LCD of 2 and 3 is 6. Change each fraction into an equivalent fraction that has 6 as its denominator.

$$\frac{w}{2} - \frac{w}{3} = \frac{3}{3}\left(\frac{w}{2}\right) - \frac{2}{2}\left(\frac{w}{3}\right)$$
$$= \frac{3w}{6} - \frac{2w}{6}$$
$$= \frac{3w - 2w}{6}$$
$$= \frac{w}{6}$$

➡ Example

If $h = \frac{y}{x - y}$, what is $h + 1$ in terms of x and y?

Solution

- Add 1 on both sides of the given equation:

$$h+1=\frac{y}{x-y}+1$$

- On the right side of the equation, replace 1 with $\frac{x-y}{x-y}$:

$$h+1=\frac{y}{x-y}+\frac{x-y}{x-y}$$

- Write the sum of the numerators over the common denominator:

$$h+1=\frac{y+x-y}{x-y}=\frac{x}{x-y}$$

> **TIP**
>
> **Reciprocal Rules**
>
> **If x and y are not 0, then**
>
> $$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$$
>
> and
>
> $$\frac{1}{x}-\frac{1}{y}=\frac{y-x}{xy}.$$

THE SUM AND DIFFERENCE OF TWO RECIPROCALS

The formulas for the sum and the difference of the reciprocals of two nonzero numbers, x and y, are worth remembering:

➡ Example

If $t=\frac{1}{r}+\frac{1}{s}$, then what is $\frac{1}{t}$ in terms of r and s?

Solution

Since $t=\frac{1}{r}+\frac{1}{s}=\frac{r+s}{rs}$, then $\frac{1}{t}=\frac{rs}{r+s}$.

SOLVING EQUATIONS WITH FRACTIONS

To solve an equation that contains fractions, eliminate the fractions by multiplying each member of the equation by the lowest common multiple of all of the denominators.

➥ Example

$$\frac{2x-9}{7}+\frac{2x}{3}=\frac{8x-1}{21}$$

What value of x is the solution of the above equation?

Solution

The smallest number that 3, 7, and 21 divide evenly into is 21. Eliminate the fractions by multiplying each member of the equation by 21:

$$\overset{3}{\cancel{21}}\left(\frac{2x-9}{\cancel{7}}\right)+\overset{7}{\cancel{21}}\left(\frac{2x}{\cancel{3}}\right)=\overset{1}{\cancel{21}}\left(\frac{8x-1}{\cancel{21}}\right)$$

$$3(2x-9)+7(2x)=8x-1$$

$$6x-27+14x=8x-1$$

$$20x-8x=27-1$$

$$12x=26$$

$$\frac{12x}{12}=\frac{26}{12}$$

$$x=\frac{13}{6}$$

LESSON 3-4 TUNE-UP EXERCISES

Multiple-Choice

1. $\frac{20b^3 - 8b}{4b}$

(A) $5b^2 - 2b$
(B) $5b^3 - 2$
(C) $5b^2 - 8b$
(D) $5b^2 - 2$

$$\frac{3}{w} - \frac{4}{3} = \frac{5w}{10w^2}$$

2. In the equation above, what is the value of w?

(A) $\frac{15}{8}$
(B) $\frac{18}{11}$
(C) $\frac{23}{12}$
(D) $\frac{13}{6}$

$$P = 4x - z + 3y$$
$$Q = -x + 4z + 3y$$

3. Using the definitions above for P and Q, what is $2P - Q$?

(A) $7x - 6z + 3y$
(B) $9x + 2z + 9y$
(C) $9x - 6z + 3y$
(D) $7x - 6z + 9y$

4. If $(x - y)^2 = 50$ and $xy = 7$, what is the value of $x^2 + y^2$?

(A) 8
(B) 36
(C) 43
(D) 64

5. If $p = \frac{a}{a-b}$ and $a \neq b$, then, in terms of a and b, $1 - p =$

(A) $\frac{a}{b-a}$

(B) $\frac{b}{b-a}$

(C) $\frac{a}{a-b}$

(D) $\frac{b}{a-b}$

6. If $(a-b)^2 + (a+b)^2 = 24$, then $a^2 + b^2 =$

(A) 4
(B) 12
(C) 16
(D) 18

$$\frac{2}{p} - \frac{1}{2p} = \frac{p^2+1}{p^2+1}$$

7. In the equation above, what is the value of $\frac{1}{p}$?

(A) $\frac{1}{3}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 3

8. If $r = t + 2$ and $s + 2 = t$, then $rs =$

(A) t^2
(B) 4
(C) $t^2 - 4$
(D) $t^2 - 4t + 4$

9. Which statement is true for all real values of x and y?

(A) $(x+y)^2 = x^2 + y^2$
(B) $x^2 + x^2 = x^4$

(C) $\frac{2^{x+2}}{2^x} = 4$
(D) $(3x)^2 = 6x^2$

10. If $(p - q)^2 = 25$ and $pq = 14$, what is the value of $(p + q)^2$?
(A) 25
(B) 36
(C) 53
(D) 81

11. If $\frac{a}{2} - \frac{b}{3} = 1$, what is $2a + 3b$ in terms of b?
(A) $\frac{7b}{3} + 1$
(B) $\frac{13b}{3} + 4$
(C) $\frac{13b + 1}{3}$
(D) $\frac{17b}{3}$

12. If $(x + 5)(x + p) = x^2 + 2x + k$, then
(A) $p = 3$ and $k = 5$
(B) $p = -3$ and $k = 15$
(C) $p = 3$ and $k = -15$
(D) $p = -3$ and $k = -15$

13. For what value of p is $(x - 2)(x + 2) = x(x - p)$?
(A) −4
(B) 0
(C) $\frac{2}{x}$
(D) $\frac{4}{x}$

$$\frac{m}{2} - \frac{3(m-4)}{5} = \frac{5(3-m)}{6}$$

14. What value of m makes the equation above a true statement?

(A) $\frac{8}{27}$

(B) $\frac{3}{22}$

(C) $\frac{62}{27}$

(D) $\frac{147}{22}$

15. If $\left(k+\frac{1}{k}\right)^2 = 16$, then $k^2+\frac{1}{k^2}$

(A) 4
(B) 8
(C) 12
(D) 14

16. If $\frac{a}{b} = 1 - \frac{x}{y}$, then $\frac{b}{a} =$

(A) $\frac{x}{y-x}$

(B) $\frac{y}{x} - 1$

(C) $\frac{y}{x-y}$

(D) $\frac{y}{y-x}$

$$\frac{11+s}{12r} = \frac{1}{6} + \frac{1-3s}{4r}$$

17. In the equation above, what is r in terms of s?

(A) $r = 5s + 4$

(B) $r = \frac{s-2}{3}$

(C) $r = 4s + 5$

(D) $r = \frac{s+3}{2}$

Grid-In

1. If $(3y-1)(2y+k) = ay^2 + by - 5$ for all values of y, what is the value of $a + b$?

2. If $4x^2 + 20x + r = (2x + s)^2$ for all values of x, what is the value of $r - s$?

$$\frac{5}{8} = \frac{-(11 - 7y)}{4y} + \frac{1}{2y} - \frac{1}{8}$$

3. What value of y makes the equation above a true statement?

LESSON 3-5 FACTORING

OVERVIEW

Factoring reverses multiplication.

Operation	Example
Multiplication	$2(x + 3y) = 2x + 6y$
Factoring	$2x + 6y = 2(x + 3y)$

There are three basic types of factoring that you need to know for the SAT:

- Factoring out a common monomial factor, as in

$$4x^2 - 6x = 2x(x - 3) \quad \text{and} \quad ay - by = y(a - b)$$

- Factoring a quadratic trinomial using the reverse of FOIL, as in

$$x^2 - x - 6 = (x - 3)(x + 2) \text{ and}$$
$$x^2 - 2x + 1 = (x - 1)(x - 1) = (x - 1)^2$$

- Factoring the difference between two squares using the rule

$$a^2 - b^2 = (a + b)(a - b)$$

FACTORING A POLYNOMIAL BY REMOVING A COMMON FACTOR

If all the terms of a polynomial have factors in common, the polynomial can be factored by using the reverse of the distributive law to remove these common factors. For example, in

$$24x^3 + 16x = 8x(3x^2 + 2)$$

$8x$ is the *G*reatest *C*ommon *F*actor (GCF) of $24x^3$ and $16x$ since 8 is the GCF of 24 and 16, and x is the greatest power of that variable that is contained in both $24x^3$ and $16x$. The factor that corresponds to $8x$ can be obtained by dividing $24x^3 + 16x$ by $8x$:

$$\frac{24x^3 + 16x}{8x} = \frac{24x^3}{8x} + \frac{16x}{8x} = 3x^2 + 2$$

You can check that the factorization is correct by multiplying $3x^2 + 2$ by $8x$ and verifying that the product is $24x^3 + 16x$.

USING FACTORING TO ISOLATE VARIABLES IN EQUATIONS

It may be necessary to use factoring to help isolate a variable in an equation in which terms involving the variable cannot be combined into a single term.

➡ Example

If $ax - c = bx + d$, what is x in terms of a, b, c, and d?

Solution

Isolate terms involving x on the same side of the equation.

- On each side of the equation add c and subtract bx:

$$ax - bx = c + d$$

- Factor out x from the left side of the equation:

$$x(a - b) = c + d$$

- Divide both sides of the new equation by the coefficient of x:

$$\frac{x\overset{1}{\cancel{(a-b)}}}{\cancel{a-b}} = \frac{c+d}{a-b}$$

$$x = \frac{c+d}{a-b}$$

FACTORING A QUADRATIC TRINOMIAL: $x^2 + bx + c$

A quadratic trinomial like $x^2 - 7x + 12$ contains x^2 as well as x. Quadratic trinomials that appear on the SAT can be factored as the product of two binomials by reversing the FOIL multiplication process.

- Think: "What two integers when multiplied together give +12 and when added together give −7?"
- Recall that, since the product of these integers is *positive* 12, the two integers must have the same sign. Hence, the integers are limited to the following pairs of factors of +12:

$$1 \text{ and } 12; -1 \text{ and } -12$$

$$2 \text{ and } 6; -2 \text{ and } -6$$

$$3 \text{ and } 4; -3 \text{ and } -4$$

- Choose −3 and −4 as the factors of 12 since they add up to −7. Thus,

$$x^2 - 7x + 12 = (x - 3)(x - 4)$$

- Use FOIL to check that the product $(x - 3)(x - 4)$ is $x^2 - 7x + 12$.

➡ Example

Factor $n^2 - 5n - 14$.

Solution

Find two integers that when multiplied together give −14 and when added together give −5. The two factors of −14 must have different signs since their product is negative. Since $(+2)(-7) = -14$ and $2 + (-7) = -5$, the factors of −14 you are looking for are +2 and −7. Thus:

$$n^2 - 5n - 14 = (n + 2)(n - 7)$$

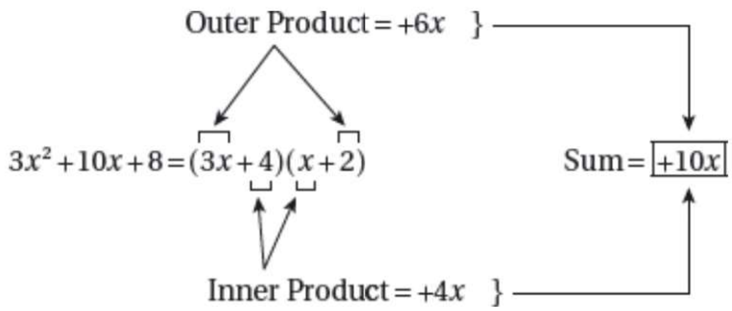
FACTORING $ax^2 + bx + c$ WHEN $a > 1$

Factoring a quadratic trinomial becomes more complicated when the coefficient of the x^2-term is greater than 1. To factor $3x + 10x + 8$:

- Factor the x^2-term and set up the binomial factors:

$$3x^2 + 10x + 8 = (3x + \boxed{?})(x + \boxed{?})$$

- Identify possibilities for the unknown pair of integers in the binomial factors. The product of these integers must be +8, the last term of $3x^2 + 10x + 8$. The possibilities are: 1 and 8; −1 and −8; 2 and 4; and −2 and −4.
- Use trial and elimination to determine the correct pair of factors of 8 and their proper placement in the binomial factors. The factors of +8 must be chosen and placed so that the sum of the outer and inner products of the terms of the binomial factors is equal to $+10x$, the middle term of $3x^2 + 10x + 8$:



- Check that the factors work. The placement of the factors of 8 matters. Although $(3x + 2)(x + 4)$ contains the correct factors of 8, they are not placed correctly since the sum of the outer and inner products is $12x + 2x = 14x$ rather than $10x$.

Factoring the Difference Between Two Squares

Since $(a + b)(a - b) = a^2 - b^2$, any binomial of the form $a^2 - b^2$ can be rewritten as $(a + b)$ times $(a - b)$.

$$a^2 - b^2 = (a + b)(a - b)$$

This means that the difference between two squares can be factored as the product of the sum and difference of the quantities that are being squared. Here are some examples in which this factoring rule is used:

- $y^2 - 16 = (y + 4)(y - 4)$
- $x^2 - 0.25 = (x + 0.5)(x - 0.5)$

- $100 - x^4 = (10 - x^2)(10 + x^2)$

➡ Example

If $\frac{x^{p^2}}{x^{q^2}} = (x^{50})^2$ with $x > 1$ and $p - q = 4$, what is the value of $p + q$?

(A) 25
(B) 50
(C) 64
(D) 96

Solution

- Since $\frac{x^{p^2}}{x^{q^2}} = x^{p^2-q^2}$ and $(x^{50})^2 = x^{100}$, $x^{p^2-q^2} = x^{100}$. Since the bases are the same, the exponents must be the same:

$$p^2 - q^2 = 100$$

- Factoring $p^2 - q^2$ makes $(p - q)(p + q) = 100$.

- Substituting 4 for $p - q$ gives $4(p + q) = 100$ so $p + q = \frac{100}{4} = 25$.

The correct choice is **(A)**.

Factoring Completely

To factor a polynomial into factors that cannot be further factored, it may be necessary to use more than one factoring technique. For example:

- $3t^2 - 75 = 3(t^2 - 25) = 3(t + 5)(t - 5)$
- $t^3 - 6t^2 + 9t = t(t^2 - 6t + 9) = t(t - 3)(t - 3)$ or $t(t - 3)^2$

Using Factoring to Simplify Algebraic Fractions

To simplify an algebraic fraction, factor the numerator and the denominator. Then cancel any factor that is in both the numerator and the denominator of the fraction since any nonzero quantity divided by itself is 1.

➡ Example

Simplify $\frac{2b-2a}{a^2-b^2}$.

Solution

- Factor the numerator and the denominator: $\frac{2b-2a}{a^2-b^2}=\frac{2(b-a)}{(a+b)(a-b)}$
- Rewrite $b-a$ as $-(a-b)$: $=\frac{2[-(a-b)]}{(a+b)(a-b)}$
- Cancel $a-b$ since it is a factor of the numerator and a factor of the denominator: $=\frac{-2\overset{1}{\cancel{(a-b)}}}{(a+b)\cancel{(a-b)}}$

$$=\frac{-2}{a+b}$$

➡ Example

$$\frac{x^2-4x-5}{(x-1)^2-4(x-1)}$$

The expression above is equivalent to

(A) $x+1$

(B) $\frac{x+5}{x-1}$

(C) $\frac{x+1}{x-1}$

(D) $\frac{x+1}{x-4}$

> **TIP**
>
> **If you get stuck on an algebraic solution, plug in an easy number. If $x = 3$, then $\frac{x^2-4x-5}{(x-1)^2-4(x-1)}$ evaluates to 2. After plugging 3 into *each* of the four answer choices, you should verify that only choice (C) evaluates to 2.**

Solution

Factor the numerator as the product of two binomials and factor out the common binomial term of $(x - 1)$ in the denominator:

$$\frac{x^2 - 4x - 5}{(x-1)^2 - 4(x-1)} = \frac{(x+1)(x-5)}{(x-1)[(x-1)-4]}$$

$$= \frac{(x+1)\overset{1}{\cancel{(x-5)}}}{(x-1)\cancel{(x-5)}}$$

$$= \frac{x+1}{x-1}$$

The correct choice is **(C)**.

➡ Example

$$\frac{3x}{2x-6} + \frac{9}{6-2x}$$

Which of the following is equivalent to the expression above?

(A) $-\frac{3}{2}$

(B) $\frac{-6x^2 + 36x - 54}{(2x-6)(6-2x)}$

(C) $\frac{3x+9}{2x-6}$

(D) $\frac{3}{2}$

Solution

$$\frac{3x}{2x-6} + \frac{9}{6-2x} = \frac{3x}{2x-6} + \frac{9}{-(2x-6)}$$

$$= \frac{3x}{2x-6} - \frac{9}{2x-6}$$

$$= \frac{3x-9}{2x-6}$$

$$= \frac{3\cancel{(x-3)}}{2\cancel{(x-3)}}$$

$$= \frac{3}{2}$$

The correct choice is **(D)**.

LESSON 3-5 TUNE-UP EXERCISES

Multiple-Choice

$$q = \frac{d}{d+n}$$

1. On a manufacturer's assembly line, d parts are found to be defective and n parts are nondefective. The formula above is used to calculate a quality-of-parts ratio. What is d expressed in terms of the other two variables?

   (A) $\frac{n}{1-q}$

   (B) $\frac{nq}{1-q}$

   (C) $\frac{n}{q-1}$

   (D) $\frac{nq}{q-1}$

2. The sum of $\frac{a}{a^2-b^2}$ and $\frac{b}{a^2-b^2}$ is

   (A) $\frac{1}{a-b}$

   (B) $\frac{a}{a-b}$

   (C) $\frac{b}{a-b}$

   (D) $\frac{a+b}{a-b}$

3. If $ax + x^2 = y^2 - ay$, what is a in terms of x and y?

   (A) $y - x$

   (B) $x - y$

   (C) $x + y$

   (D) $\frac{x^2+y^2}{x-y}$

4. If $\frac{xy}{x+y} = 1$ and $x \neq -y$, what is x in terms of y?

(A) $\frac{y+1}{y-1}$

(B) $\frac{y+1}{y}$

(C) $\frac{y}{y-1}$

(D) $\frac{y}{y+1}$

5. What is the sum of $\frac{4x}{x-1}$ and $\frac{4x+4}{x^2-1}$, expressed in simplest form?

(A) $\frac{4x+1}{x-1}$

(B) $\frac{4(x+1)}{x-1}$

(C) $\frac{4(x^2+4x+1)}{x^2-1}$

(D) $\frac{4(x+2)}{x^2-1}$

6. If $h = \frac{x^2-1}{x+1} + \frac{x^2-1}{x-1}$, what is x in terms of h?

(A) $\frac{h}{2}$

(B) $2h + 1$

(C) $2h - 1$

(D) $\sqrt{\frac{h}{2}}$

7. If $ax^2 - bx = ay^2 + by$, then $\frac{a}{b} =$

(A) $\frac{1}{x-y}$

(B) $\frac{1}{x+y}$

(C) $\frac{x-y}{x+y}$

(D) $\frac{x+y}{x-y}$

8. If $a \neq b$ and $\frac{a^2-b^2}{9} = a + b$, then what is the value of $a - b$?

(A) $\frac{1}{2}$

(B) 3
(C) 9
(D) 12

9. If $\frac{r+s}{x-y}=\frac{3}{4}$, then $\frac{8r+8s}{15x-15y}=$

(A) $\frac{32}{45}$

(B) $\frac{8}{15}$

(C) $\frac{7}{16}$

(D) $\frac{2}{5}$

10. If $x^2 = k + 1$, then $\frac{x^4-1}{x^2+1}=$

(A) k
(B) k^2
(C) $k+2$
(D) $k-2$

11. If $p = x(3x + 5) - 28$, then p is divisible by which of the following expressions?

(A) $3x + 4$
(B) $x - 4$
(C) $x + 7$
(D) $3x - 7$

12. If $(x + p)$ is a factor of both $x^2 + 16x + 64$ and $4x^2 + 37x + k$, where p and k are nonzero integer constants, what could be the value of k?

(A) 9
(B) 24
(C) 40
(D) 63

LESSON 3-6 QUADRATIC EQUATIONS

OVERVIEW

A ***quadratic equation*** is an equation in which the greatest exponent of the variable is 2, as in $x^2 + 3x - 10 = 0$. A quadratic equation has two roots, which can be found by breaking down the quadratic equation into two first-degree equations.

ZERO-PRODUCT RULE

If the product of two or more numbers is 0, at least one of these numbers is 0.

➡ Example

For what values of x is $(x - 1)(x + 3) = 0$?

Solution

Since $(x - 1)(x + 3) = 0$, either $x - 1 = 0$ or $x + 3 = 0$.

- If $x - 1 = 0$, then $x = 1$.
- If $x + 3 = 0$, then $x = -3$.

The possible values of x are 1 and -3.

SOLVING A QUADRATIC EQUATION BY FACTORING

The two roots of a quadratic equation may or may not be equal. For example, the equation $(x - 3)^2 = 0$ has a double root of $x = 3$. The quadratic equation $x^2 = 9$, however, has two unequal roots, $x = 3$ and $x = -3$. More complicated quadratic equations on the SAT can be solved by factoring the quadratic expression.

➡ Example

Solve $x^2 + 2x = 0$ for x.

Solution

- Factor the left side of the quadratic equation: $x^2 + 2x = 0$
  $x(x + 2) = 0$
- Form two first-degree equations by setting each factor equal to 0: $x = 0$ or $x + 2 = 0$
- Solve each first-degree equation: $x = 0$ or $x = -2$

  The two roots are 0 and −2.

If a quadratic equation does not have all its nonzero terms on the same side of the equation, you must put the equation into this form before factoring.

➡ Example

Solve $x^2 + 3x = 10$ for x.

Solution

To rewrite the quadratic equation so that all the nonzero terms are on the same side:

- Subtract 10 from both sides of $x^2 + 3x = 10$: $x^2 + 3x - 10 = 0$
- Factor the quadratic polynomial: $(x + 5)(x - 2) = 0$
- Set each factor equal to 0: $x + 5 = 0$ or $x - 2 = 0$
- Solve each equation: $x = -5$ or $x = 2$

The two roots are −5 and 2.

You can check that $x = -5$ and $x = 2$ are the roots by plugging each value into $x^2 + 3x = 10$ and verifying that the left side then equals 10, the right side.

➡ Example

$$2y^2 = 3(y + 9)$$

If j and k represent solutions of the equation above and $j > k$, what is the value of $j - k$?

Solution

- Rewrite the equation in standard $ay^2 + by + c = 0$ form:

$$2y^2 - 3(y + 9) = 0$$

$$2y^2 - 3y - 27 = 0$$

- Solve by factoring:

$$(2y + ?)(y + ?) = 0 \leftarrow \text{Missing terms are the factors of } -27$$

$$(2y - 9)(y + 3) = 0 \leftarrow \text{Verify the factors and their placement by multiplying out using } FOIL$$

$$2y - 9 = 0 \quad or \quad y + 3 = 0$$

$$y = \frac{9}{2} \quad or \quad y = -3$$

- Set $j = \frac{9}{2}$ and $k = -3$ so

$$\begin{aligned} j - k &= \frac{9}{2} - (-3) \\ &= \frac{9}{2} + 3 \\ &= \frac{9}{2} + \frac{6}{2} \\ &= \frac{15}{2} \end{aligned}$$

➥ Example

If 4 is a root of $x^2 - x - w = 0$, what is the value of w?

Solution

Since 4 is a root of the given equation, replacing x with 4 in that equation gives an equation that can be used to solve for w:

$$\begin{aligned} 4^2 - 4 - w &= 0 \\ 16 - 4 - w &= 0 \\ 12 - w &= 0 \\ 12 &= w \end{aligned}$$

Hence, w = **12**.

QUADRATIC EQUATIONS WITH EQUAL ROOTS

A quadratic equation may have equal roots. If $x^2 - 2x + 1 = 0$, then

$$(x - 1)(x - 1) = 0$$

$$x - 1 = 0 \quad or \quad x - 1 = 0$$

$$x = 1 \qquad\qquad x = 1$$

Thus, the equation $x^2 - 2x + 1 = 0$ has a double root of **1**.

TIP

Every quadratic equation has two roots, but the roots may be equal.

LESSON 3-6 TUNE-UP EXERCISES

Multiple-Choice

1. The fraction $\frac{x-2}{x^2+4x-21}$ is *not* defined when $x =$

   (A) 2
   (B) 7 or −3
   (C) −7 or 3
   (D) −7 or −3

2. If $\frac{a^2}{2} = 2a$, then a equals

   (A) 0 or −2
   (B) 0 or 2
   (C) 0 or −4
   (D) 0 or 4

3. If $(s-3)^2 = 0$, what is the value of $(s+3)(s+5)$?

   (A) 48
   (B) 24
   (C) 15
   (D) 0

4. If $k = 7 + \frac{8}{k}$, what is the value of $k^2 + \frac{64}{k^2}$?

   (A) 33
   (B) 49
   (C) 64
   (D) 65

$$\frac{18-3w}{w+6} = \frac{w^2}{w+6}$$

5. Which of the following represents the *sum* of all possible solutions to the equation above?

   (A) −9
   (B) −3
   (C) 3

(D) 9

$$\text{Equation (1): } 2x^2 + 7x = 4$$
$$\text{Equation (2): } (y - 1)^2 = 9$$

6. If f is the greater of the two roots of Equation (1) and g is the lesser of the two roots of Equation (2), what is the value of the product $f \times g$?
   (A) −4
   (B) −1
   (C) 2
   (D) 8

$$x^3 - 20x = x^2$$

7. If a, b, and c represent the set of all values of x that satisfy the equation above, what is the value of $(a + b + c) + (abc)$?
   (A) −1
   (B) 0
   (C) 1
   (D) 9

8. If $\frac{x^2}{3} = x$, then $x =$
   (A) 0 or −3
   (B) 3 only
   (C) 0 only
   (D) 0 or 3

9. By how much does the sum of the roots of the equation $(x + 1)(x - 3) = 0$ exceed the product of its roots?
   (A) 1
   (B) 2
   (C) 3
   (D) 5

10. If $x^2 - 63x - 64 = 0$ and p and n are integers such that $p^n = x$, which of the following CANNOT be a value for p?
   (A) −8

(B) −4
(C) −1
(D) 4

11. If $r > 0$ and $r^t = 6.25r^{t+2}$, then $r =$

(A) $\frac{2}{5}$

(B) $\frac{4}{9}$

(C) $\frac{5}{8}$

(D) $\frac{3}{4}$

$$\frac{x}{2x-1} = \frac{2x+1}{x+2}$$

12. If m and n represent the solutions of the equation above, what is the value of $m + n$?

(A) $-\frac{4}{3}$

(B) $-\frac{3}{4}$

(C) $\frac{2}{3}$

(D) $\frac{5}{4}$

$$\frac{1}{(t-2)^2} = 6 + \frac{1}{(t-2)}$$

13. If p and q represent the solutions of the equation above, what is the value of $p \times q$?

(A) $-\frac{3}{2}$

(B) $\frac{7}{2}$

(C) $\frac{9}{4}$

(D) $\frac{15}{8}$

Grid-In

1. If $(4p + 1)^2 = 81$ and $p > 0$, what is a possible value of p?

2. If $(x - 1)(x - 3) = -1$, what is a possible value of x?

3. By what amount does the sum of the roots exceed the product of the roots of the equation $(x - 5)(x + 2) = 0$?

$$(3k + 14)k = 5$$

4. If r and s represent the solutions of the equation above and $r > s$, what is the value of the difference $r - s$?

$$x^4 + 16 = 10x^2$$

5. If p and q are distinct roots of the equation above and $pq > 0$, what is the value of the product pq?

$$(2a - 5)^2 = (4 - 3a)^2$$

6. What is the sum of the roots of the equation above?

LESSON 3-7 SYSTEMS OF EQUATIONS

OVERVIEW

A **system of equations** is a set of equations whose solution makes each of the equations true at the same time. SAT questions involving systems of two linear equations with two different variables can usually be solved by

- Substituting the solution of one equation into the other equation to eliminate one of the variables in that equation; or
- Adding or subtracting corresponding sides of the two equations so that an equation with only one variable results.

SOLVING A LINEAR SYSTEM BY SUBSTITUTION

To solve a linear system using the substitution method, pick the equation in which it is easy to solve for one of the variables. After solving the equation for that variable, plug the solution into the other equation. In the linear system,

$$2y - 3x = 29$$
$$y + 2x = 4$$

it is easy to solve the second equation for *y*, which gives $y = -2x + 4$. Plug the solution into the first equation to eliminate *y*:

$$\begin{aligned} 2\overbrace{(-2x+4)}^{y} - 3x &= 29 \\ -4x + 8 - 3x &= 29 \\ -7x + 8 &= 29 \\ -7x &= 21 \\ \frac{-7x}{-7} &= \frac{21}{-7} \\ x &= -3 \end{aligned}$$

To find the corresponding value for y, substitute −3 for x in the simpler of the two original equations:

$$y + 2\overbrace{(-3)}^{x} = 4$$
$$y - 6 = 4$$
$$y = 10$$

The solution is **(−3, 10)**.

➡ Example

An automobile repair shop wants to mix a solution that is 35% pure antifreeze with another solution that is 75% pure antifreeze. How many liters of each solution must be used in order to produce 80 liters of solution that is 50% pure antifreeze?

Solution

Use x and y to represent the number of liters of the 35% and 75% solutions, respectively, in the mixture. The problems has two conditions:

> **CONDITION 1:** The mixture will contain 80 liters of solution: $x + y = 80$.
>
> **CONDITION 2:** The sum of the number of liters of pure antifreeze in the two ingredients must be equal to the number of liters of pure antifreeze in the mixture:

$$0.35x + 0.75y = 0.50(80) = 40$$

- Represent the two conditions by a system of linear equations:

$$x + y = 80$$
$$0.35x + 0.75y = 40$$

- From the first equation, $y = 80 - x$. Eliminate y in the second equation:

$$0.35x + 0.75(80 - x) = 40$$
$$0.35x + 60 - 0.75x = 40$$
$$-0.40x = 40 - 60$$
$$\frac{-0.40x}{-0.40} = \frac{-20}{-0.40}$$
$$x = 50$$

- Find y by substituting 50 for x in the first equation:

$$50 + y = 80 \text{ so } y = 80 - 50 = 30$$

Thus, **50 liters** of the 35% solution and **30 liters** of the 75% solution must be used.

NOTE: The problem could be solved using one variable where x and $80 - x$ represent the number of liters of the 35% and 75% solutions, respectively. This means that $0.35x + 0.75(80 - x) = 40$, which can be solved as before.

SOLVING A LINEAR SYSTEM BY COMBINING EQUATIONS

Sometimes the easiest way of solving a system of linear equations is to rewrite each equation in the standard $Ax + By = C$ form (where A, B, and C are constants). If the same variable in both equations have the opposite (or the same) numerical coefficients, then adding (or subtracting) corresponding sides of the two equations will eliminate one of the variables. In the system

$$x - 2y = 5$$
$$x + 2y = 11$$

the numerical coefficients of y are opposites so adding corresponding sides of the two equations will eliminate it:

$$x - 2y = 5$$
$$\underline{x + 2y = 11}$$
$$2x + 0 = 16$$
$$x = \frac{16}{2} = 8$$

Find y by substituting 8 for x in either of the original equations:

$$8 + 2y = 11$$
$$2y = 3$$
$$y = \frac{3}{2}$$

➡ Example

$$3x = 5y + 19$$
$$2x + y = -x + 7$$

What ordered pair (x, y) represents the solution to the system of equations above?

Solution

- Rewrite each equation in standard $Ax + By = C$ form:

$$\begin{aligned} 3x - 5y &= 19 \\ \underline{3x + \ \ y} &\underline{= \ 7} \end{aligned}$$

- Eliminate x by subtracting the second equation from the first equation:

$$\begin{aligned} 3x - 5y &= 19 \\ -\underline{3x + \ y} &\underline{= \ 7} \\ 0 - 6y &= 12 \\ y &= \frac{12}{-6} = -2 \end{aligned}$$

- Solve for x in the first equation by substituting –2 for y:

$$3x = 5(-2) + 19$$
$$3x = -10 + 19$$
$$3x = 9$$
$$x = \frac{9}{3} = 3$$

The solution is **(3, −2)**.

SOLVING A LINEAR SYSTEM BY USING MULTIPLIERS

Before combining corresponding sides of the equations in a linear system, it may be necessary to first multiply one or both equations by a number that makes the same variable in both equations have opposite numerical coefficients.

➡ Example

$$2x + 4y = 5x + 11$$
$$6x - 5y = -16$$

What ordered pair (x, y) represents the solution to the system of equations above?

Solution

- Write the first equation in the standard form $Ax + By = C$:

$$\begin{aligned} -3x + 4y &= 11 \\ 6x - 5y &= -16 \end{aligned}$$

- Change the coefficient of x in the first equation to −6 by multiplying the equation by 2. Then add the resulting equation and the second equation thereby eliminating x:

$$\left.\begin{aligned} (2)[-3x+4y=11] \\ 6x-5y=-16 \end{aligned}\right\} \longrightarrow \begin{aligned} -6x+8y &= 22 \\ +\ 6x-5y &= -16 \\ \hline 0+3y &= 6 \\ y = \frac{6}{3} &= 2 \end{aligned}$$

- Find x by substituting 2 for y in either of the original equations:

$$\begin{aligned} 6x - 5y &= -16 \\ 6x - 5(2) &= -16 \\ 6x &= -16 + 10 \\ 6x &= -6 \\ \frac{6x}{6} &= \frac{-6}{6} \\ x &= -1 \end{aligned}$$

The solution is **(−1, 2)**.

➡ Example

$$\begin{aligned} 2x + 5y &= 2y - 6 \\ 5x + 2y &= 7 \end{aligned}$$

In the system of equations above, what is the value of the product *xy*?

Solution

Solve the system to get the values of *x* and *y*. Rewrite the first equation in *Ax* + *By* = *C* form:

$$\begin{aligned} 2x + 3y &= -6 \\ 5x + 2y &= \;\;7 \end{aligned}$$

- Eliminate *y* by multiplying the first equation by −2, multiplying the second equation by 3, and then adding corresponding sides of the two resulting equations:

$$\begin{aligned} -2[2x + 3y = -6] \;\Rightarrow\; & -4x - 6y = 12 \\ 3[5x + 2y = 7] \;\Rightarrow\; & \underline{15x + 6y = 21} \\ & 11x + 0 = 33 \\ & x = \frac{33}{11} = 3 \end{aligned}$$

- Find *y* by substituting 3 for *x* in either of the two original equations:

$$5(3) + 2y = 7$$
$$15 + 2y = 7$$
$$2y = -8$$
$$y = \frac{-8}{2} = -4$$

■ Since the solution is (3, −4), $xy = (3)(-4) = \mathbf{-12}$.

➡ Example

A store that offers faxing services charges a fixed amount to fax one page and a different amount for faxing each additional page. If the cost of faxing 5 pages is $3.05 and the cost of faxing 13 pages is $6.65,

a. What is the cost of faxing one page?

b. What is the total cost of faxing three pages to the same telephone number?

Solution

a. If x represents the cost of faxing the initial page and y the cost of faxing each additional page, then

$$x + 12y = 6.65$$
$$x + 4y \ = 3.05$$

Solve for x by multiplying the second equation by −3 and then adding the result to the first equation:

$$\begin{array}{r} x + 12y = 6.65 \\ \underline{-3x - 12y = -9.15} \\ -2x + 0 \ = -2.50 \\ \frac{-2x}{-2} = \frac{-2.50}{-2} \\ x = 1.25 \end{array}$$

The cost of faxing one page is **$1.25**.

b. Find the cost of faxing each additional page after the first page.

- Solve for y in the second equation:

$$\begin{aligned} 1.25 + 4y &= 3.05 \\ 4y &= 3.05 - 1.25 \\ \frac{4y}{4y} &= \frac{1.80}{4} \\ y &= 0.45 \end{aligned}$$

- The cost of faxing 3 pages is the sum of the charges for the first page and the two additional pages:

$$\begin{aligned} x + 2y &= 1.25 + 2(0.45) \\ &= 1.25 + 0.90 \\ &= 2.15 \end{aligned}$$

The cost of faxing 3 pages is **$2.15**.

SOLVING OTHER TYPES OF SYSTEMS OF EQUATIONS

If a system of equations has more variables than equations, you may be asked to solve for some combination of letters.

➡ Example

If $2r = s$ and $24t = 3s$, what is r in terms of t?

Solution

Since the question asks for r in terms of t, work toward eliminating s.

- Substitute $2r$ for s in the second equation:

$$24t = 3s = 3(2r) = 6r$$

- Solve for r in the equation $24t = 6r$:

$$\begin{aligned} \frac{24t}{6} &= \frac{6r}{6} \\ 4t &= r \end{aligned}$$

Hence, $r = \mathbf{4}t$.

➡ Example

If $ab - 3 = 12$ and $2bc = 5$, what is the value of $\frac{a}{c}$?

Solution

Since the question asks for $\frac{a}{c}$, you must eliminate b.

- Find the value of ab in the first equation. Since $ab - 3 = 12$, then $ab = 15$.
- To eliminate b, divide corresponding sides of $ab = 15$ and $2bc = 5$:

$$\frac{ab}{2bc} = \frac{15}{5}$$

$$\frac{ab}{2bc} = 3$$

$$\frac{a}{2c} = 3$$

Solve the resulting equation for $\frac{a}{c}$:

$$2\left(\frac{a}{2c}\right) = 2(3)$$

$$\frac{a}{c} = 6$$

The value of $\frac{a}{c}$ is **6**.

LESSON 3-7 TUNE-UP EXERCISES

Multiple-Choice

1. If $2x - 3y = 11$ and $3x + 15 = 0$, what is the value of y?

   (A) -7
   (B) -5
   (C) $\frac{1}{3}$
   (D) 3

2. If $2a = 3b$ and $4a + b = 21$, then $b =$

   (A) 1
   (B) 3
   (C) 4
   (D) 7

3. If $2p + q = 11$ and $p + 2q = 13$, then $p + q =$

   (A) 6
   (B) 8
   (C) 9
   (D) 12

$$2(x + y) = 3y + 5$$
$$3x + 2y = -3$$

4. Which equivalent equation could be used to solve the system of equations above?

   (A) $3\left(\frac{5+y}{2}\right) + 2y = -3$
   (B) $3\left(\frac{5}{2} - y\right) + 2y = -3$
   (C) $3x + 2(2x - 5) = -3$
   (D) $3x + 2(5 - 2x) = -3$

5. If $x - y = 3$ and $x + y = 5$, what is the value of y?

   (A) -4
   (B) -2

(C) −1
(D) 1

6. If $5x + y = 19$ and $x - 3y = 7$, then $x + y =$

(A) −4
(B) −1
(C) 3
(D) 4

7. If $x - 9 = 2y$ and $x + 3 = 5y$, what is the value of x?

(A) −2
(B) 4
(C) 11
(D) 17

8. If $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$ and $\frac{1}{x} - \frac{1}{y} = \frac{3}{4}$, then $x =$

(A) $\frac{1}{4}$
(B) $\frac{1}{2}$
(C) 2
(D) 4

9. If $5a + 3b = 35$ and $\frac{a}{b} = \frac{2}{5}$, what is the value of a?

(A) $\frac{14}{5}$
(B) $\frac{7}{2}$
(C) 5
(D) 7

10. If $\frac{x}{y} = 6$, $\frac{y}{w} = 4$, and $x = 36$, what is the value of w?

(A) $\frac{1}{2}$
(B) $\frac{3}{2}$
(C) 2
(D) 4

11. If $4r + 7s = 23$ and $r - 2s = 17$ then $3r + 3s =$

(A) 8
(B) 24
(C) 32
(D) 40

12. If $\frac{p-q}{2} = 3$ and $rp - rq = 12$, then $r =$

(A) −1
(B) 1
(C) 2
(D) 4

13. If $(a + b)^2 = 9$ and $(a - b)^2 = 49$, what is the value of $a^2 + b^2$?

(A) 17
(B) 20
(C) 29
(D) 58

$$3x - y = 8 - x$$

$$6x + 4y = 2y - 9$$

14. For the system of equations above, what is the value of the product xy?

(A) –3
(B) –2
(C) 2
(D) 3

15. If $3x + y = c$ and $x + y = b$, what is the value of x in terms of c and b?

(A) $\frac{c-b}{3}$
(B) $\frac{c-b}{2}$
(C) $\frac{b-c}{3}$
(D) $\frac{b-c}{2}$

16. If $a + b = 11$ and $a - b = 7$, then $ab =$

(A) 6
(B) 8
(C) 10
(D) 18

$$x - z = 7$$

$$x + y = 3$$

$$z - y = 6$$

17. For the above system of three equations, $x =$

(A) 5
(B) 6
(C) 7
(D) 8

$$a = 4c$$

$$c = re$$

$$a = 5e$$

18. For the system of equations above, if $e \neq 0$, what is the value of r?

(A) $\frac{1}{20}$

(B) $\frac{4}{5}$

(C) $\frac{5}{4}$

(D) 1

19. During the next football season, a player's earnings, x, will be 0.005 million dollars more than those of a teammates' earnings, y. The two players will earn a total of 3.95 milion dollars. Which system of equations could be used to determine the amount each player will earn, in millions of dollars?

(A) $x + y = 3.95$
$x + 0.005 = y$

(B) $x - 3.95 = y$
$y + 0.005 = x$

(C) $y - 3.95 = x$
$x + 0.005 = y$

(D) $x + y = 3.95$
$y + 0.005 = x$

Food	Protein	Calories
Cereal	5 g	90
Milk	8 g	80

20. The table above shows the number of grams of protein and the number of calories in single servings of bran flakes cereal and milk. How many servings of each are needed to get a total of 35 grams of protein and 470 calories?

(A) 2 servings of milk; 4 servings of cereal

(B) $2\frac{1}{2}$ servings of milk; $2\frac{1}{2}$ servings of cereal

(C) 3 servings of milk; $2\frac{1}{2}$ servings of cereal

(D) $2\frac{1}{2}$ servings of milk; 3 servings of cereal

Grid-In

1. If 5 sips + 4 gulps = 1 glass and 13 sips + 7 gulps = 2 glasses, how many sips equal a gulp?

2. When Amy exercises in her fitness center for 1 hour she burns a total of 475 calories. If she burns 9 calories a minute jogging on the treadmill and then burns 6.5 calories a minute pedaling on the stationary bicycle, how many minutes of the hour does she spend exercising on the bicycle?

3. John and Sara each bought the same type of pen and notebook in the school bookstore, which does not charge sales tax. John paid $5.55 for

two pens and three notebooks, and Sara paid \$3.50 for one pen and two notebooks. How much does the school bookstore charge for one notebook?

$$\frac{1}{2}r - \frac{1}{3}s = 8$$

$$\frac{5}{8}r - \frac{1}{4}s = 29$$

4. For the system of equations above, what is the value of $r + s$?

5. During its first week of business, a market sold a total of 108 apples and oranges. The second week, five times the number of apples and three times the number of oranges were sold. A total of 452 apples and oranges were sold during the second week. How many more apples than oranges were sold in the *first* week?

6. Jacob and Zachary go to the movie theater and purchase refreshments for their friends. Jacob spends a total of \$18.25 on two bags of popcorn and three drinks. Zachary spends a total of \$27.50 for four bags of popcorn and two drinks. What is the cost for purchasing one bag of popcorn and one drink?

For **Questions 7 and 8** refer to the information below.

A mobile phone–based taxi service charges a base fee of \$2 plus an amount per minute and an additional amount per mile for the trip. Ariel is charged \$16.94 for a ride that takes 14 minutes and travels 10 miles. Victoria is charged \$11.30 for a ride that takes 10 minutes and travels 6 miles.

7. What is the per-minute charge?

8. What would be the charge for a ride that takes 8 minutes and travels 5 miles?

LESSON 3-8 ALGEBRAIC INEQUALITIES

OVERVIEW

A linear inequality such as $2x - 3 \le 7$ is solved in much the same way that a linear equation is solved. There is one exception: multiplying or dividing both sides of an inequality by the same negative quantity *reverses* the direction of the inequality. For example, if $-3x \le 12$, then dividing both sides by -3 results in the equivalent inequality, $x \ge -4$.

"AT LEAST" AND "AT MOST"

The phrase "is at least" is translated by $\ge$ and the phrase "is at most" is translated by $\le$.

➡ Example

What is the greatest integer value of x such that $1 - 2x$ is *at least* 6?

Solution

$$1 - 2x \ge 6$$
$$-2x \ge 5$$
$$\frac{-2x}{-2} \le \frac{5}{-2} \leftarrow \text{Reverse inequality sign}$$
$$x \le -2.5$$

The greatest integer value of x that satisfies the inequality is **–3**.

➡ Example

$$C = 8n + 522$$

The equation above gives the cost, C, in dollars of manufacturing n items. A profit is made when the total revenue from selling a quantity of items is greater than the total cost of manufacturing the same quantity of items. If each item can be sold for $14, which of the following inequalities gives all possible values of n that will produce a profit?

(A) $n > 87$
(B) $n > 112$
(C) $n > 522$
(D) $n > 609$

Solution

Since a profit will be made when total revenue is greater than total cost,

$$14n > 8n + 522$$
$$14n - 8n > 522$$
$$6n > 522$$
$$\frac{6n}{6} > \frac{522}{6}$$
$$n > 87$$

The correct choice is **(A)**.

➡ Example

If $2x^2 + 3ax - 7 < -17$, what is the smallest possible integer value of a when $x = -1$?

Solution

Replace x with -1 and solve for a:

$$2(-1)^2 + 3a(-1) - 7 < -17$$
$$2 - 3a - 7 < -17$$
$$-3a - 5 < -17$$
$$-3a < -12$$
$$\frac{-3a}{-3} > \frac{-12}{-3} \leftarrow \text{Reverse inequality sign}$$
$$a > 4$$

The smallest integer value of a that satisfies the inequality is **5**.

SOLVING COMBINED INEQUALITIES

To solve an inequality that has the form $c \leq ax + b \leq d$, where a, b, c, and d stand for numbers, isolate the letter by performing the same operation on each member of the inequality.

➡ Example

If $-2 \leq 3x - 7 \leq 8$, find x.

Solution

- Add 7 to each member of the inequality:

$$-2 \leq 3x - 7 \leq 8$$
$$-2 + 7 \leq 3x - 7 + 7 \leq 8 + 7$$
$$5 \leq 3x \leq 15$$

- Divide each member of the inequality by 3:

$$\frac{5}{3} \leq \frac{3x}{3} \leq \frac{15}{3}$$
$$\frac{5}{3} \leq x \leq 5$$

The solution consists of all real numbers greater than or equal to $\frac{5}{3}$ and less than or equal to 5:

$$\frac{5}{3} \leq x \leq 5$$

➡ Example

If $3 < x + 1 < 8$ and $2 < y < 9$, which of the following best describes the range of values of $y - x$?

(A) $-7 < y - x < 5$
(B) $-5 < y - x < 7$
(C) $0 < y - x < 2$
(D) $2 < y - x < 7$

Solution

First find the upper and lower limits of x and y. If $3 < x + 1 < 8$, then $3 - 1 < x < 8 - 1$ so $2 < x < 7$. This means that the lower limit of x is 2 and the upper limit is 7. Since $2 < y < 9$, the lower limit of y is 2 and the upper limit is 9.

- The upper limit of $y - x$ is obtained by taking the difference between the upper limit of y and the lower limit of x which is $9 - 2 = 7$.
- The lower limit of $y - x$ is obtained by taking the difference between the lower limit of y and the upper limit of x which is $2 - 7 = -5$.
- Hence, $-5 < y - x < 7$.

The correct choice is **(B)**.

TIP

The direction of the inequality sign also gets reversed when comparing the reciprocals of two positive numbers x and y:

If $x > y$ then $\frac{1}{x} < \frac{1}{y}$

ORDERING PROPERTIES OF INEQUALITIES

- If $a < b$ and $b < c$, then $a < c$. For example:

$$2 < 3 \text{ and } 3 < 4, \text{ so } 2 < 4$$

- If $a < b$ and $x < y$, then $a + x < b + y$. For example:

$$2 < 3 \text{ and } 4 < 5, \text{ so } 2 + 4 < 3 + 5$$

- If $a < b$ and $x > y$, then the relationship between $a + x$ and $b + y$ cannot be determined until each of the variables is replaced by a specific number.

LESSON 3-8 TUNE-UP EXERCISES

Multiple-Choice

1. What is the largest integer value of p that satisfies the inequality $4 + 3p < p + 1$?

   (A) -2
   (B) -1
   (C) 0
   (D) 1

2. If $-3 < 2x + 5 < 9$, which of the following CANNOT be a possible value of x?

   (A) -2
   (B) -1
   (C) 0
   (D) 2

3. Roger is having a picnic for 78 guests. He plans to serve each guest at least one hot dog. If each package, p, contains eight hot dogs, which inequality could be used to determine the number of packages of hot dogs Roger must buy?

   (A) $\frac{p}{8} \geq 78$
   (B) $8p \geq 78$
   (C) $8 + p \geq 78$
   (D) $78 - p \geq 8$

4. Peter begins his kindergarten year able to spell 10 words. He is going to learn to spell 2 new words every day. Which inequality can be used to determine how many days, d, it takes Peter to be able to spell at least 85 words?

   (A) $2d + 10 \geq 85$
   (B) $20d \leq 85$
   (C) $(d + 2) + 10 \geq 85$
   (D) $2d - 10 \leq 85$

5. Which of the following numbers is NOT a solution of the inequality $7 - 5x \leq -3(x-5)$?

(A) –5
(B) –4
(C) –2
(D) 1

6. Tamara has a cell phone plan that charges \$0.07 per minute plus a monthly fee of \$19.00. She budgets \$29.50 per month for total cell phone expenses without taxes. What is the maximum number of minutes Tamara could use her phone each month in order to stay within her budget?

(A) 150
(B) 271
(C) 421
(D) 692

7. What is the solution of $3(2m - 1) \leq 4m + 7$?

(A) $m \geq 5$
(B) $m \leq 5$
(C) $m \geq 4$
(D) $m \leq 4$

8. An online music club has a one-time registration fee of \$13.95 and charges \$0.49 to buy each song. If Emma has \$50.00 to join the club and buy songs, what is the maximum number of songs she can buy?

(A) 73
(B) 74
(C) 130
(D) 131

9. The ninth grade class at a local high school needs to purchase a park permit for \$250.00 for their upcoming class picnic. Each ninth grader attending the picnic pays \$0.75. Each guest pays \$1.25. If 200 ninth graders attend the picnic, which inequality can be used to determine the number of guests, x, needed to cover the cost of the permit?

(A) $0.75x - (1.25)(200) \geq 250.00$

(B) $0.75x + (1.25)(200) \geq 250.00$
(C) $(0.75)(200) - 1.25x \geq 250.00$
(D) $(0.75)(200) + 1.25x \geq 250.00$

10. If $2(x-4) \geq \frac{1}{2}(5-3x)$ and x is an integer, what is the smallest possible value of x^2?

(A) $\frac{1}{4}$
(B) 1
(C) 4
(D) 9

11. Edith tutors after school for which she gets paid at a rate of \$20 an hour. She has also accepted a job as a library assistant that pays \$15 an hour. She will work both jobs, but she is able to work *no more than* a total of 11 hours a week, due to school commitments. Edith wants to earn *at least* \$185 a week working a combination of both jobs. Which inequality can be used to represent the situation?

(A) $20(11+x)+\frac{185}{x}>15$
(B) $20x+15(11-x)>185$
(C) $15(11-x)+\frac{185}{x}>20$
(D) $15x+20(11+x)>185$

12. Guy is paid \$185 per week plus 3% of his total sales in dollars, and Jim is paid \$275 per week plus 2.5% of his total sales in dollars. If d represents the dollar amount of sales for each person, which inequality represents the amount of sales for which Guy is paid more than Jim?

(A) $d > 18{,}000$
(B) $d < 18{,}000$
(C) $d > 12{,}500$
(D) $d < 12{,}500$

13. Connor wants to attend the town carnival. The price of admission to the carnival is \$4.50, and each ride costs an additional 79 cents. If he can spend at most \$16.00 at the carnival, which inequality can be used to

solve for r, the number of rides Connor can go on, and what is the maximum number of rides he can go on?

(A) $0.79 + 4.50r \le 16.00$; 3 rides
(B) $0.79 + 4.50r \le 16.00$; 4 rides
(C) $4.50 + 0.79r \le 16.00$; 14 rides
(D) $4.50 + 0.79r \le 16.00$; 15 rides

14. For how many integer values of b is $b + 3 > 0$ and $1 > 2b - 9$?

(A) Four
(B) Five
(C) Six
(D) Seven

Grid-In

1. For what integer value of y is $y + 5 > 8$ and $2y - 3 < 7$?

2. If 2 times an integer x is increased by 5, the result is always greater than 16 and less than 29. What is the least value of x?

3. If $2 < 20x - 13 < 3$, what is one possible value for x?

$$\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\frac{1}{10}<\frac{1}{8}-\frac{1}{9}+\frac{1}{10}+\frac{1}{n}$$

4. For the above inequality, what is the greatest possible positive integer value of n?

5. Chelsea has \$45 to spend at an amusement park. She spends \$20 on admission and \$15 on snacks. She wants to play a game that costs \$0.65 per game. What is the maximum number of times she can play the game?

6. Chris rents a booth at a flea market at a cost of \$75 for one day. At the flea market Chris sells picture frames each of which costs him \$6.00. If Chris sells each picture frame for \$13, how many picture frames must he sell to make a profit of *at least* \$200 for that day?

7. An online electronics store must sell at least \$2,500 worth of printers and monitors per day. Each printer costs \$125 and each monitor costs

$225. The store can ship a maximum of 15 items per day. What is the maximum number of printers it can ship each day?

$$-\frac{5}{3} < \frac{1}{2} - \frac{1}{3}x < -\frac{3}{2}$$

8. For the inequality above, what is a possible value of $x - 3$?

LESSON 3-9 ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

OVERVIEW

The absolute value of x, written as $|x|$, refers to the quantity x without regard to whether it is positive or negative. Geometrically, $|x|$ represents the distance from 0 to x on the number line. Since −2 and +2 are each 2 units from 0, $|2| = 2$ and $|-2| = 2$.

SOLVING ABSOLUTE VALUE EQUATIONS: $|x - a| = b$

To solve an absolute value equation, remove the absolute value sign by accounting for two possibilities:

- The quantity inside the absolute value sign is nonnegative in which case simply remove the absolute value sign:

$$\text{If } x - a \geq 0, \text{ then } |x - a| = x - a.$$

- The quantity inside the absolute value sign is negative in which case remove the absolute value sign by placing a negative sign in front of the quantity:

$$\text{If } x - a < 0, \text{ then } |x - a| = -(x - a).$$

➥ Example

Solve for x: $|x - 1| = 4$.

Solution

Consider the two possibilities:

- If the quantity inside the absolute value sign is nonnegative, then

$$|x - 1| = x - 1 = 4, \text{ so } x = 5$$

- If the quantity inside the absolute value sign is negative, then

$$\begin{aligned} |x - 1| = -(x - 1) &= 4 \\ -x + 1 &= 4 \\ -x &= 3 \\ x &= -3 \end{aligned}$$

- The two possible solutions for x are **5** or **−3**. You should verify that both roots satisfy the original absolute value equation.

➡ Example

Solve for x: $|2x + 3| + 4 = 5$.

> **TIP**
>
> **If c is a negative number, there is *no* value of x that makes $|x| = c$ or $|x| < c$ true.**

Solution

If $|2x + 3| + 4 = 5$, then $|2x + 3| = 1$. Thus,

- $2x + 3 = 1$, so $2x = 1 - 3 = -2$ and $x = \frac{-2}{2} = -1$; or
- $2x + 3 = -1$, so $2x = -1 - 3 = -4$ and $x = \frac{-4}{2} = -2$

You should verify that both roots satisfy the original equation.

➡ Example

Solve and check: $|x - 3| = 2x$.

Solution

If $|x - 3| = 2x$, then

- $x - 3 = 2x$ so $-3 = x$.
  Check: If $x = -3$, then

$$|-3 - 3| = 2(-3)$$
$$|-6| \neq -6$$

- $x - 3 = -2x$ so $3x - 3 = 0$, $3x = 3$ and $x = 1$.
  Check: If $x = 1$, then

$$|x - 3| = 2x$$
$$|1 - 3| = 2(1)$$
$$|-2| = 2 \checkmark$$

Hence, $x = \mathbf{1}$ is the only root of the absolute value equation.

INTERPRETING ABSOLUTE VALUE INEQUALITIES

The absolute value inequality $|x - a| < d$ represents the set of all points x that are less than d units from a. For example,

- the inequality $|t - 68°| < 3°$ states that the temperature, t, is less than 3° from 68°, which means that t is between 65° and 71°, as shown in Figure 3.2.

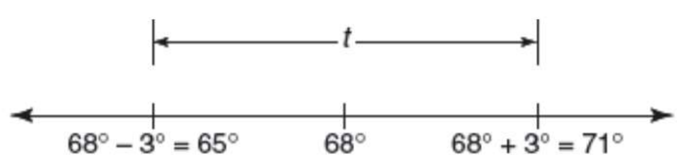


Figure 3.2 Solution of $|t - 68°| < 3°$

- the inequality $|t - 68°| > 3°$ states that the temperature, t, is more than 3° from 68°, which means that t is less than 65° or greater than 71°.

SOLVING ABSOLUTE VALUE INEQUALITIES

To solve an absolute value inequality algebraically, remove the absolute value sign according to the following rules where d is a positive number:

- if $|ax - b| < d$, then $-d < ax - b < d$.
- if $|ax - b| > d$, then $ax - b < -d$ or $ax - b > d$.

➡ Example

Solve and graph the solution set of $|2x - 1| \leq 7$.

Solution

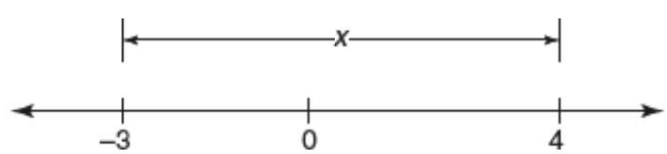
- If $|2x - 1| \leq 7$, then $-7 \leq 2x - 1 \leq 7$.
- Add 1 to each member of the combined inequality:

$$\begin{array}{rcl} -7 \leq & 2x - 1 & \leq 7 \\ +1 & +1 & +1 \\ \hline -6 \leq & 2x & \leq 8 \end{array}$$

- Divide each member of the combined inequality by 2:

$$\frac{-6}{2} \leq \frac{2x}{2} \leq \frac{8}{2}$$
$$-3 \leq x \leq 4$$

- Graph the solution set:



➡ Example

If $5 < |2 - x| < 6$ and $x > 0$, what is one possible value of x?

Solution

Remove the absolute value sign:

- If $2 - x \geq 0$, then $5 < 2 - x < 6$, so $3 < -x < 4$. Multiplying each term of the inequality by -1 gives $-3 > x > -4$. Since it is given that $x > 0$, disregard this solution.

- If $2 - x < 0$, then $5 < -(2 - x) < 6$ or, equivalently, $5 < x - 2 < 6$, so $5 + 2 < x < 6 + 2$ and $7 < x < 8$.
- Therefore, x can be any number between 7 and 8, such as **7.5**.

LESSON 3-9 TUNE-UP EXERCISES

Multiple-Choice

$$|n - 1| < 4$$

1. How many integers n satisfy the inequality above?
   (A) Two
   (B) Five
   (C) Seven
   (D) Nine

2. If $|x| \leq 2$ and $|y| \leq 1$, then what is the least possible value of $x - y$?
   (A) -3
   (B) -2
   (C) -1
   (D) 0

3. If $\left|\frac{1}{2}x\right| \geq \frac{1}{2}$, then which statement must be true?
   (A) $x \leq -2$ or $x \geq 2$
   (B) $x \leq -1$ or $x \geq 1$
   (C) $x \leq -\frac{1}{2}$ or $x \geq \frac{1}{2}$
   (D) $-1 \leq x \leq 1$

4. If $\frac{1}{2}|x|$ and $|y| = x + 1$, then y^2 could be
   (A) 2
   (B) 3
   (C) 4
   (D) 9

5. In a certain greenhouse for plants, the Fahrenheit temperature, F, is controlled so that it does *not* vary from 79° by more than 7°. Which of the following best expresses the possible range in Fahrenheit temperatures of the greenhouse?
   (A) $|F - 79| \leq 7$

(B) $|F - 79| > 7$
(C) $|F - 7| \leq 79$
(D) $|F - 7| > 79$

6. If $\frac{|a+3|}{2} = 1$ and $2|b + 1| = 6$, then $|a + b|$ could equal any of the following EXCEPT

(A) 1
(B) 3
(C) 5
(D) 7

7. For what value of x is $|1 + x| = |1 - x|$?

(A) No value
(B) 1
(C) −1
(D) 0

$$-1 < x < 3$$

8. The inequality above is equivalent to which of the following?

(A) $|x - 1| < 2$
(B) $|x + 1| < 2$
(C) $|x - 2| < 1$
(D) $|x + 2| < 1$

9. A certain medication must be stored at a temperature, t, that may range between a low of 45° Fahrenheit and a high of 85° Fahrenheit. Which inequality represents the allowable range of Fahrenheit temperatures?

(A) $|t - 65| \leq 20$
(B) $|t + 20| \leq 65$
(C) $|t + 65| \leq 20$
(D) $|t - 20| \leq 85$

10. The inequality $|1.5C - 24| \leq 30$ represents the range of monthly average temperatures, C, in degrees Celsius, during the winter months for a certain city. What was the lowest monthly average temperature, in degrees Celsius, for this city?

(A) –4
(B) 0
(C) 6
(D) 9

Grid-In

$$|t - 7| = 4$$

$$|9 - t| = 2$$

1. What value of t satisfies both of the above equations?

2. If $|-3y + 2| < 1$, what is one possible value of y?

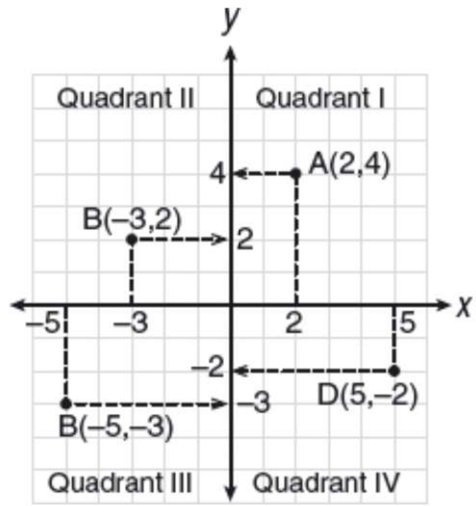
3. If $|x - 16| \leq 4$ and $|y - 6| \leq 2$, what is the greatest possible value of $x - y$?

4. An ocean depth finder shows the number of feet in the depth of water at a certain place. The difference between d, the actual depth of the water, and the depth finder reading, x, is $|d - x|$ and must be less than or equal to $0.05d$. If the depth finder reading is 620 feet, what is the *maximum* value of the actual depth of the water, to the *nearest* foot?

LESSON 3-10 GRAPHING IN THE *xy*-PLANE

OVERVIEW

The ***xy*-plane** is the plane formed by two fixed perpendicular lines, called **axes**, intersecting at their 0 points, called the **origin**. Each point in the *xy*-plane can be uniquely represented by an ordered pair of signed numbers of the form (x, y), called **coordinates**, that represents its distance from the horizontal ***x*-axis** and the vertical ***y*-axis**. The xy-plane is divided into four quadrants:



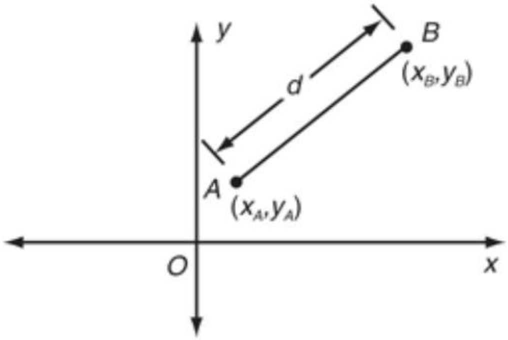
FINDING THE DISTANCE BETWEEN TWO POINTS

The distance formula can be used to find the *distance* between any two points in the *xy*-plane, say *A* and *B*, which is also the length of segment *AB*.

<table>
<tr><th>Distance Formula</th><th>Example</th></tr>
<tr><td>

The distance d between points $A(x_A, y_A)$ and $B(x_B, y_B)$ is

$$d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$



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To find the distance d between points (4, –1) and (7, 5), let $(x_A, y_A) = (4, -1)$ and $(x_B, y_B) = (7, 5)$:

$$\begin{aligned} d &= \sqrt{(7-4)^2 + (5-(-1))^2} \\ &= \sqrt{3^2 + 6^2} \\ &= \sqrt{45} \\ &= \sqrt{9} \cdot \sqrt{5} \\ &= 3\sqrt{5} \end{aligned}$$

</td></tr>
</table>

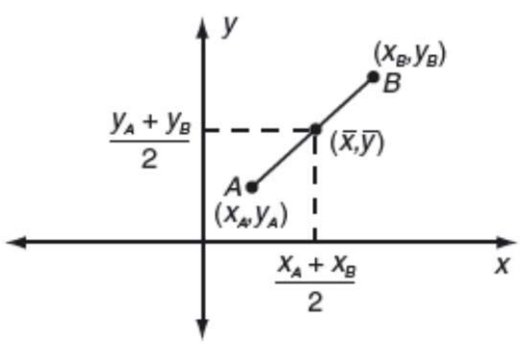
FINDING THE MIDPOINT OF A SEGMENT

The coordinates of the *midpoint* of a segment are the averages of the corresponding coordinates of the endpoints of the segment.

<table>
<tr><th>Midpoint Formula</th><th>Example</th></tr>
<tr><td>

The coordinates $(\overline{x}, \overline{y})$ of the midpoint of the segment whose endpoints are $A(x_A, y_A)$ and $B(x_B, y_B)$ are

$$\overline{x} = \frac{x_A + x_B}{2} \text{ and } \overline{y} = \frac{y_A + y_B}{2}$$



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To find the midpoint of a segment whose endpoints are (4, –1) and (7, 5), let $(x_A, y_A) =$ (4, –1) and $(x_B, y_B) = (7, 5)$. Since

$$\overline{x} = \frac{4+7}{2} = \frac{11}{2}$$

and

$$\overline{y} = \frac{(-1)+5}{2} = 2$$

the midpoint is $\left(\frac{11}{2}, 2\right)$.

</td></tr>
</table>

FINDING THE SLOPE OF A LINE

The **slope** of a line is a number that represents its steepness. It is calculated by finding the difference of the *y*-coordinates of any two points on the line (Δy), and dividing it by the difference of the *x*-coordinates of those two points (Δx) taken in the same order. The letter *m* is commonly used to represent slope.

Slope Formula	Example
The slope, m, of a nonvertical line that contains $A(x_A, y_A)$ and $B(x_B, y_B)$ is $m = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$ The order in which the coordinates of the points are subtracted must be the same in the numerator and in the denominator.	To find the slope of the line that contains points (4, –1) and (7, 5), let $(x_A, y_A) = (4, -1)$ and $(x_B, y_B) = (7, 5)$. Then: $\text{Slope} = \frac{y_B - y_A}{x_B - x_A} = \frac{5-(-1)}{7-4}$ $= \frac{5+1}{3}$ $= 2$

MATH REFERENCE FACT

If you know the slope and the coordinates of one point on a line, you can find the coordinates of other points on the same line. For example, if the slope of a line is 2 and the point (–1, 3) is on the line, then each time *x* increases by 1, *y* increases by 2 so there is another point on the line (–1 + 1, 3 + 2) = (0, 5).

POSITIVE, NEGATIVE, AND ZERO SLOPE

- A line that rises as *x* increases has a positive slope. If the line falls as *x* increases, the line has a negative slope. See Figure 3.3 where the letter *m* is used to represent the slope of a line.

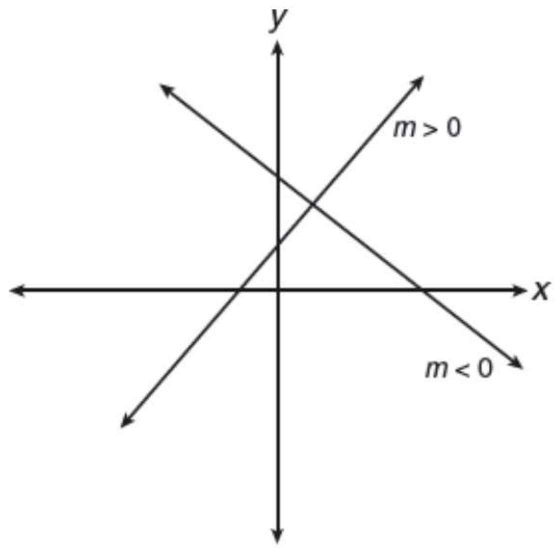


Figure 3.3 Positive vs. negative slope

- The slope of a horizontal line is 0 and the slope of a vertical line is undefined. See Figure 3.4.

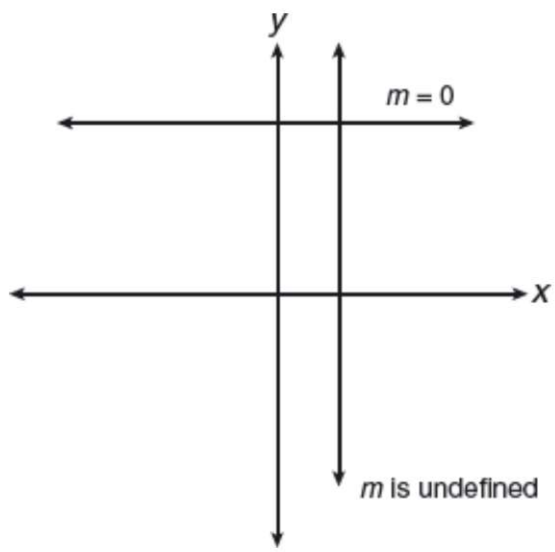


Figure 3.4 Undefined vs. 0 slope

SLOPES OF PARALLEL AND PERPENDICULAR LINES

- Parallel lines have the *same* slope. See Figure 3.5.

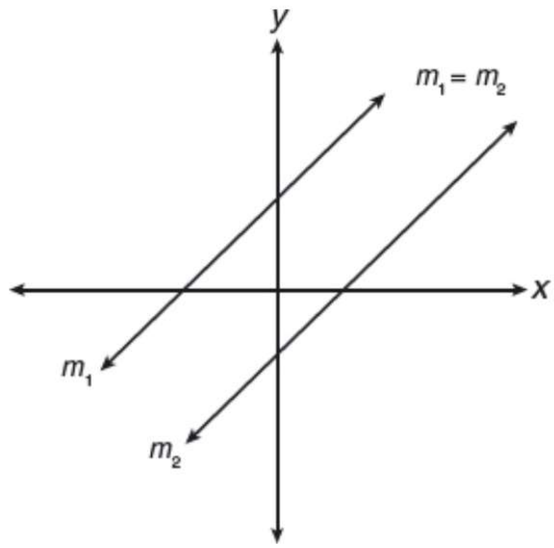


Figure 3.5 Slopes of parallel lines

- Perpendicular lines have slopes that are *negative reciprocals*. See Figure 3.6. If the slope of a line is $\frac{3}{4}$, then the slope of a perpendicular line is $\frac{3}{4}$. The product of the slopes of perpendicular lines is –1.

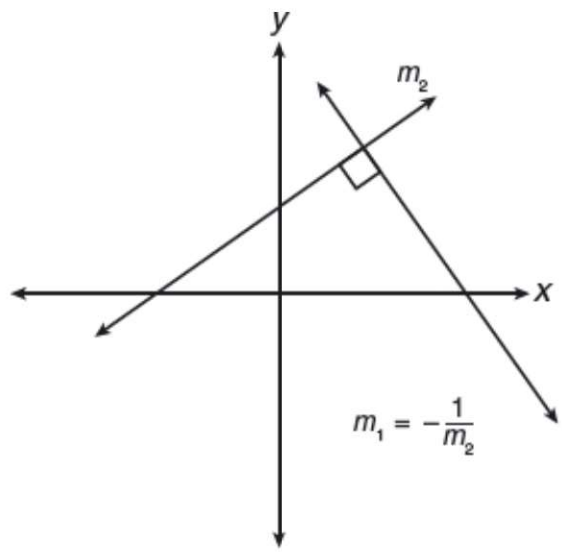


Figure 3.6 Slopes of perpendicular lines

EQUATION OF A LINE: $y = mx + b$

If a nonvertical line has a slope of m and intersects the y-axis at $(0, b)$, then the equation $y = mx + b$ describes the set of all points (x, y) that the line contains. For example:

- If an equation of a line is $y = -5x + 3$, then $m = -5$ and $b = 3$, so the slope of the line is −5 and the line crosses the y-axis at (0, 3).
- If an equation of a line is $y = 2x - 4$, then $m = 2$. The slope of a line parallel to the given line is also 2.
- If an equation of a line is $3x + y = 7$, then solving for y gives $y = -3x + 7$, so $m = -3$ and $b = 7$. The slope of a line perpendicular to the given line is the negative reciprocal of −3, which is $\frac{1}{3}$.

TIP

- **If a line that passes through the origin contains the point (*a, b*), then the slope of the line is $\frac{b}{a}$.**
- **If the slope of line *l* is *m*, then the slope of a line parallel to *l* is also *m*, and the slope of a line perpendicular to *l* is $\frac{1}{m}$, provided $m \neq 0$.**

➥ Example

Line p contains the points (–1, 8) and (9, k). If line p is parallel to line q whose equation is $3x + 4y = 7$, what is the value of k?

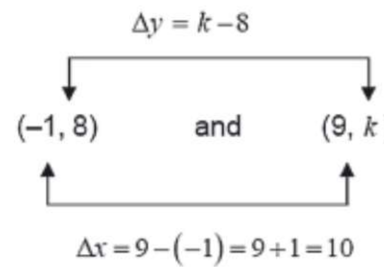
Solution

- Find the slope of $3x + 4y = 7$ by changing the equation to slope-intercept form:

$$3x + 4y = 7$$
$$4y = -3x + 7$$
$$y = -\frac{3}{4}x + \frac{7}{4}$$

  The slope of line q is $-\frac{3}{4}$.
- Using the slope formula, represent the slope of line p in terms of k:

$$m = \frac{y_B - y_A}{x_B - x_A}$$
$$= \frac{k-8}{10}$$



- Since parallel lines have the same slope, set the slope of line p equal to the slope of line q:

$$\frac{k-8}{10} = \frac{-3}{4}$$
$$4(k-8) = -3(10)$$
$$4k - 32 = -30$$
$$4k = -30 + 32$$
$$\frac{4k}{4} = \frac{2}{4}$$
$$k = \frac{1}{2}$$

WRITING AN EQUATION OF A LINE

The y-coordinate of the point at which a line crosses the y-axis is called the ***y*-intercept** of the line. If you know the slope (m) and the y-intercept (b) of a line, you can form its equation by writing $y = mx + b$.

- Suppose you know that the slope of a line is 2 and that the line contains the point (−1, 1). Since it is given that $m = 2$, the equation of the line must look like $y = 2x + b$. Because the line contains the point (−1, 1), when $x = -1$, $y = 1$:

$$y = 2x + b$$
$$1 = 2(-1) + b$$
$$1 = -2 + b$$
$$3 = b$$

Hence, an equation of this line is $y = 2x + 3$.

- If you know that the line contains the points (4, 0) and (−1, 5), you can find its equation by first determining the slope of the line. If Δx represents the difference in the x-coordinates of the two points and Δy stands for the difference in the y-coordinates of these points, then

$$m = \frac{\Delta y}{\Delta x} = \frac{5-0}{-1-4} = \frac{5}{-5} = -1$$

Since $m = -1$, the equation of the line must look like $y = -x + b$. To find the y-intercept, b, substitute the coordinates of either given point into the equation. Since $y = 0$ when $x = 4$:

$$\begin{aligned} y &= -x + b \\ 0 &= -4 + b \\ 4 &= b \end{aligned}$$

Hence, an equation of the line is $y = -x + 4$.

WRITING AN EQUATION OF A LINE FROM ITS GRAPH

You can determine the y-intercept and the slope of a line from its graph.

> **TIP**
>
> **Memorize the formulas for midpoint, distance, and slope, as these formulas are *not* provided in the math reference section of the actual test. You are expected to be able to recall and apply these formulas, including in problems that involve forming or analyzing the equation of a line.**

- To figure out the slope of the line shown in Figure 3.7, form a right triangle by moving 1 unit to the right from any point on the line, say (0, 3), and then moving up or down until the line is reached. The horizontal side of the right triangle is formed by moving 1 unit to the *right*, so $\Delta x = +1$. The vertical side of the right triangle is formed by moving 4 units *up*, so $\Delta y = +4$. Hence, the slope of the line is

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{1} = 4$$

Since $m = 4$ and $b = 3$, an equation of the line is $y = 4x + 3$.

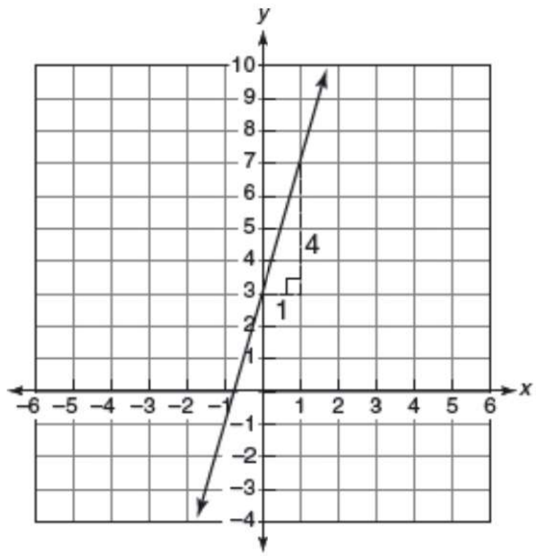


Figure 3.7 Determining the equation $y = 4x + 3$ from its graph

- To figure out the slope of the line shown in Figure 3.8, form a right triangle by moving 1 unit to the right from any point on the line, say (0, 6), and then moving up or down until the line is reached. The horizontal side of the right triangle is formed by moving 1 unit to the *right*, so $\Delta x = +1$. The vertical side of the right triangle is formed by moving 3 units *down*, so $\Delta y = -3$. Hence, the slope of the line is

$$m = \frac{\Delta y}{\Delta x} = \frac{-3}{1} = -3$$

Since $m = -3$ and $b = 6$, an equation of the line is $y = -3x + 6$.

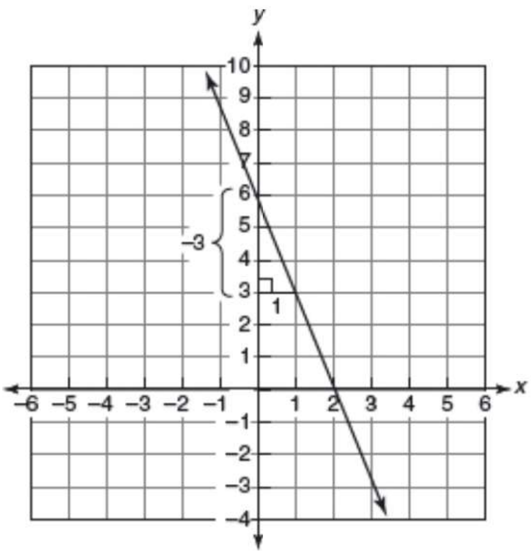
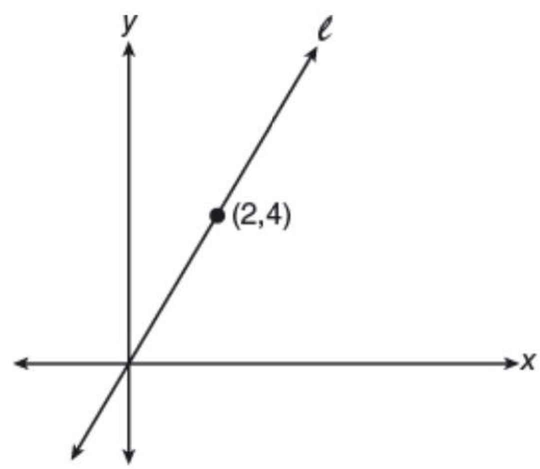


Figure 3.8 Determining the equation $y = -3x + 6$ from its graph

➡ Example



In the accompanying figure, line ℓ passes through the origin and the point (2, 4). Line m (not shown) is perpendicular to line ℓ at (2, 4). Line m intersects the x-axis at which point?

(A) (5, 0)
(B) (6, 0)
(C) (8, 0)
(D) (10, 0)

Solution

The slope of line ℓ is $\frac{4-0}{2-0} = 2$.

- Since the slopes of perpendicular lines are negative reciprocals, the slope of line *m* is $-\frac{1}{2}$.

  The equation of line *m* has the form $y = -\frac{1}{2}x + b$.
- Because line *m* contains the point (2, 4), the coordinates of this point must satisfy its equation:

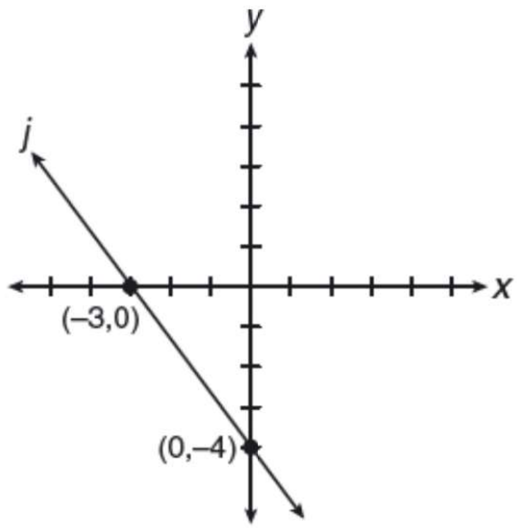
$$y = -\frac{1}{2}x + b$$
$$4 = -\frac{1}{2}(2) + b$$
$$4 = -1 + b$$
$$5 = b$$

  An equation of line *m* is $y = -\frac{1}{2}x + 5$.
- Line *m* intersects the *x*-axis where $y = 0$:

$$0 = -\frac{1}{2}x + 5$$
$$-5 = -\frac{1}{2}(x)$$
$$(-2)(-5) = (-2)\left(-\frac{1}{2}x\right)$$
$$10 = x$$

Since line *m* intersects the *x*-axis at (10, 0), the correct choice is **(D)**.

➥ Example

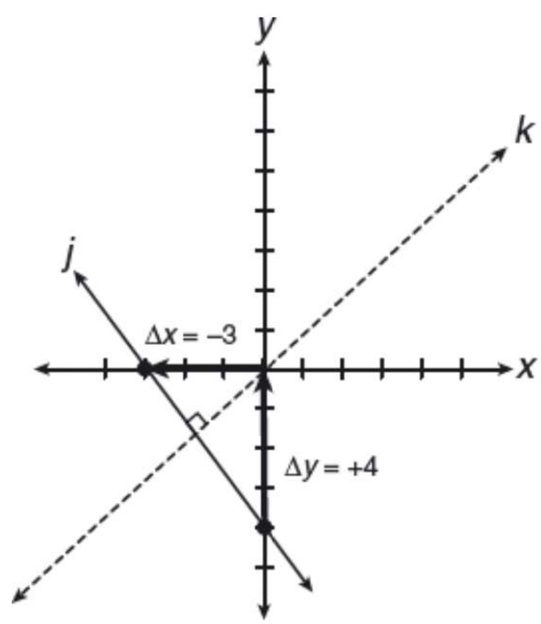


In the figure above, line k (not shown) is perpendicular to line j. If the equation of line k is $y = px$, what is the value of the constant p?

Solution

Use the diagram to determine the slope of line j.

- From the y-intercept to the x-intercept, the vertical "rise" is +4 and the horizontal "run" is –3:



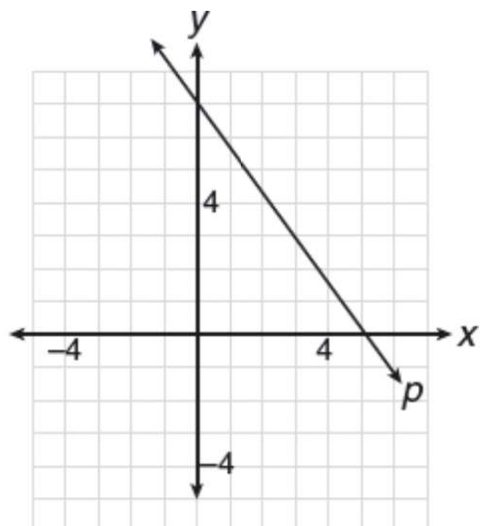
- The slope of line j is "rise" over "run":

$$\frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{+4}{-3} = -\frac{4}{3}$$

- Since the slopes of perpendicular lines are negative reciprocals, the slope of line k is $\frac{4}{3}$. In the equation $y = px$, p represents the slope of line k so $p = \frac{3}{4}$.

Grid-in **3/4**

➡ Example



Line p is graphed in the xy-plane above. If line p is translated up 3 units and right 5 units, what is the slope of the resulting line?

(A) $-\frac{4}{3}$

(B) $-\frac{11}{15}$

(C) $-\frac{3}{5}$

(D) $-\frac{1}{2}$

Solution

First find the slope of line p by choosing two points on the line with integer coordinates, say (2, 4) and (5, 0). Using these points, the slope of line p is

$$\frac{0-4}{5-2}=-\frac{4}{3}$$

Translating a line shifts all the points on the line the same distance and in the same direction so that the resulting line will be parallel to the original line and, as a result, have the same slope. Hence, the slope of the translated line is $-\frac{4}{3}$.

The correct choice is **(A)**.

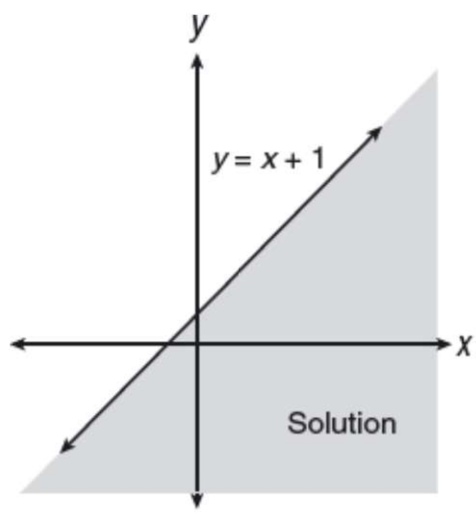
SYSTEMS OF LINEAR INEQUALITIES

The solution set for a system of two linear equations whose graphs intersect is a single point. The solution set for a system of two linear inequalities is the region over which the solution sets for the two linear inequalities overlap.

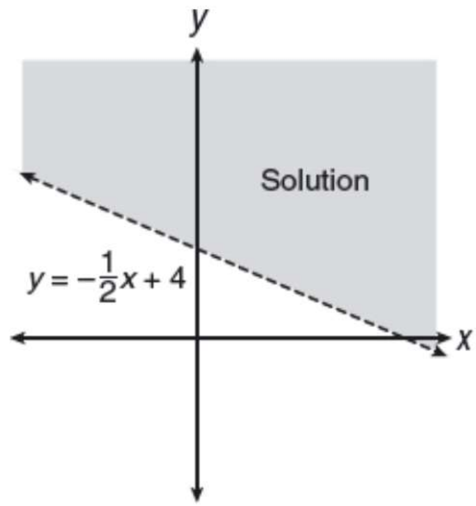
Consider the system

$$y \le x+1$$
$$y > -\frac{1}{2}x+4$$

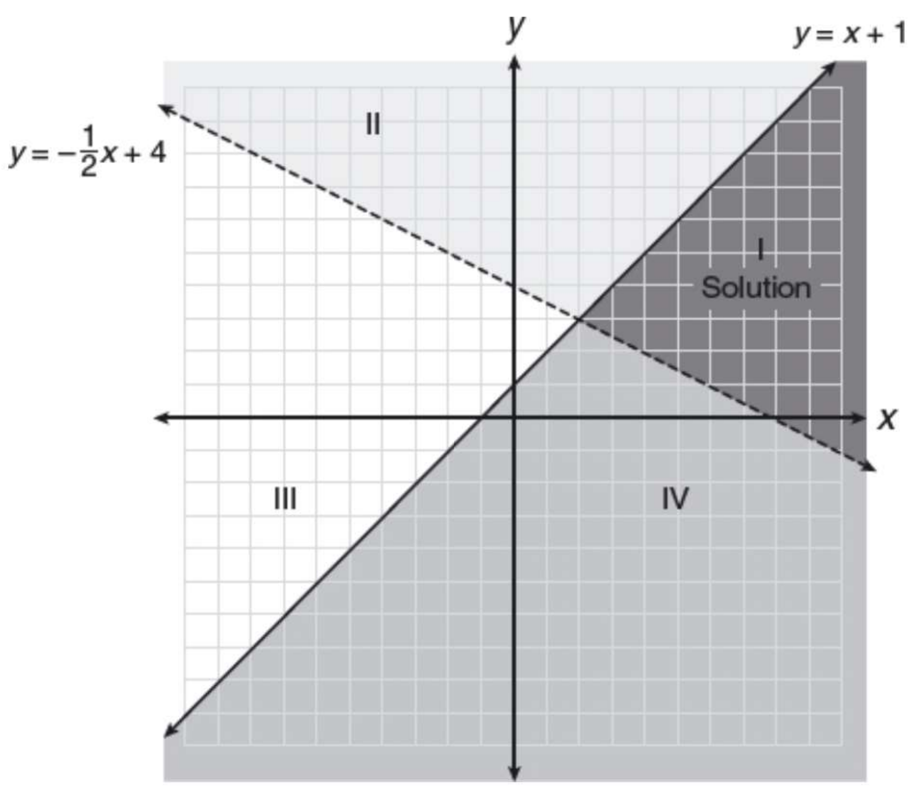
- The solution set for $y \le x + 1$ consists of the set of points that lie *below* (≤) or on the boundary line $y = x + 1$:



- The solution set for $y \ge -\frac{1}{2}x + 4$ consists of the set of all points that lie *above* (≥) or on the boundary line $y = -\frac{1}{2}x + 4$:



- The solution set for the system lies in the region (labeled I) where the two solution sets overlap:



You can also determine which of the four regions above represents the solution set to the system of linear inequalities by picking a point in each region and testing whether it makes both inequalities true at the same time. For example, test the point (6, 2), which lies in region I:

Test (6, 2): $y \le x + 1$

$$2 \le 6 + 1$$

$$2 \le 7 \qquad \textit{True!}$$

Test (6, 2): $y \ge -x + 4$

$$2 \ge -(6) + 4$$

$$2 \ge -2 \qquad \textit{True!}$$

Hence, the bounded region that contains (6, 2) represents the solution set.

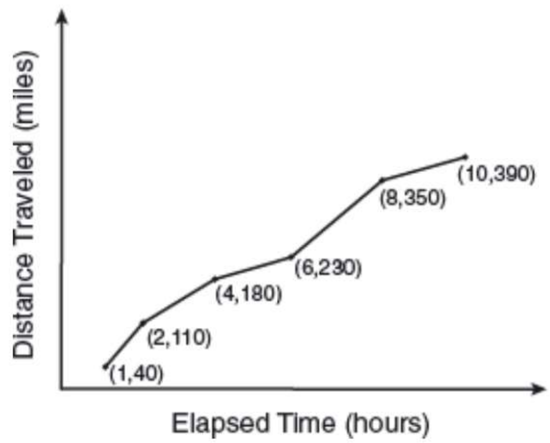
MATH FACT

To graph a linear inequality:

- Replace the inequality relation with an "=" sign and graph the boundary line. Draw a solid line for a ≥ or ≤ inequality relation and a broken line if the inequality relation is > or <.
- If after solving for y in terms of x, the inequality relation is ≥ or >, then the solution region lies *above* the boundary line; if the inequality relation is ≤ or <, the solution region lies *below* the boundary line.

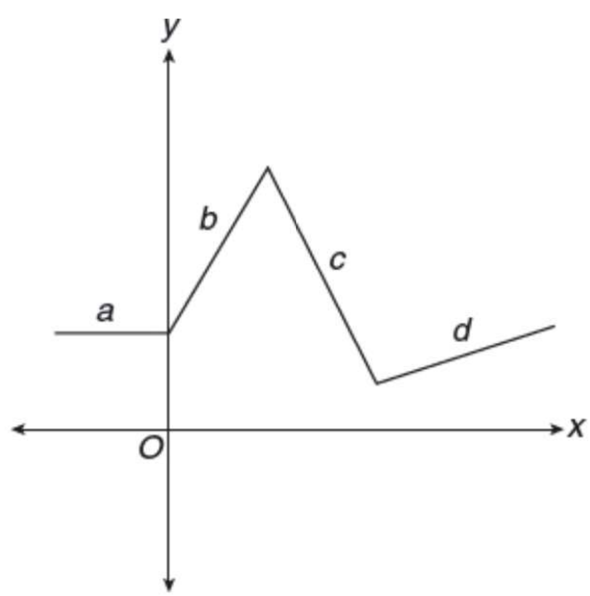
LESSON 3-10 TUNE-UP EXERCISES

Multiple-Choice



1. A family kept a log of the distance they traveled during a trip, as represented by the graph above in which the points are ordered pairs of the form (hour, distance). During which interval was their average speed the greatest?

   (A) The first hour to the second hour
   (B) The second hour to the fourth hour
   (C) The sixth hour to the eighth hour
   (D) The eighth hour to the tenth hour

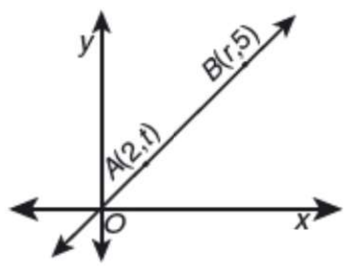


2. In the above figure, each line segment is labeled with a variable that represents the numerical value of its slope. Which inequality statement must be true?

(A) $a < c < b < d$
(B) $d < b < a < c$
(C) $d < c < b < a$
(D) $c < a < d < b$

3. Which of the following represents an equation of the line that is the perpendicular bisector of the segment whose endpoints are (–2, 4) and (8, 4)?

(A) $x = 3$
(B) $y = 3$
(C) $x = 5$
(D) $y = 5$



4. In the graph above, what is r in terms of t?

(A) $\frac{5}{2}t$
(B) $\frac{2}{5}t$
(C) $\frac{t}{10}$
(D) $\frac{10}{t}$

5. What is the slope of the line $2(x + 2y) = 0$?

(A) $\frac{1}{2}$
(B) -2
(C) $-\frac{1}{2}$

(D) 0

6. Segments *AP* and *BP* have the same length. If the coordinates of *A* and *P* are (−1, 0) and (4, 12), respectively, which could be the coordinates of *B*?

I. $\left(\frac{3}{2}, 6\right)$

II. (9, 24)

III. (−8, 7)

(A) I and II only
(B) II and III only
(C) II only
(D) III only

7. Which of the following is an equation of a line that is parallel to the line $\frac{1}{2}y - \frac{2}{3}x = 6$ in the *xy*-plane?

(A) $y = -\frac{3}{4}x + 1$

(B) $y = 4\left(\frac{x-1}{3}\right)$

(C) $9x - 6y = 18$

(D) $\frac{y}{3} = \frac{x-5}{4}$

8. Which of the following is an equation of a line that is perpendicular to the line $y = -2(x + 1)$?

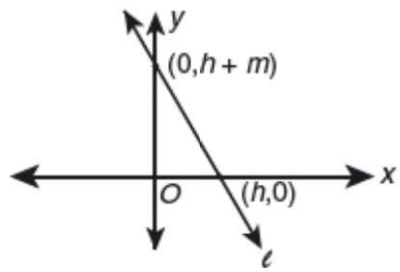
(A) $x + 2y = 7$
(B) $8x - 4y = 9$
(C) $\frac{x-1}{6} = \frac{y}{3}$
(D) $y - 2x = 0$

9. The point whose coordinates are (4, −2) lies on a line whose slope is $\frac{3}{2}$. Which of the following are the coordinates of another point on this line?

(A) (1, 0)
(B) (2, 1)
(C) (6, 1)
(D) (7, 0)

10. If point $E(5, h)$ is on the line that contains $A(0, 1)$ and $B(-2, -1)$, what is the value of h?

(A) -1
(B) 0
(C) 1
(D) 6



11. In the figure above, if the slope of line ℓ is m, what is m in terms of h?

(A) $\frac{h}{1+h}$
(B) $\frac{-h}{1+h}$
(C) $\frac{h}{1-h}$
(D) $1+h$

12. Which could be the slope of a line that contains (1, 1) and passes between the points (0, 2) and (0, 3)?

(A) $-\frac{3}{2}$
(B) $-\frac{1}{2}$
(C) 0
(D) $\frac{1}{2}$

13. The line $y + 2x = b$ is perpendicular to a line that passes through the origin. If the two lines intersect at the point $(k + 2, 2k)$, what is the value of k?

(A) $-\frac{3}{2}$

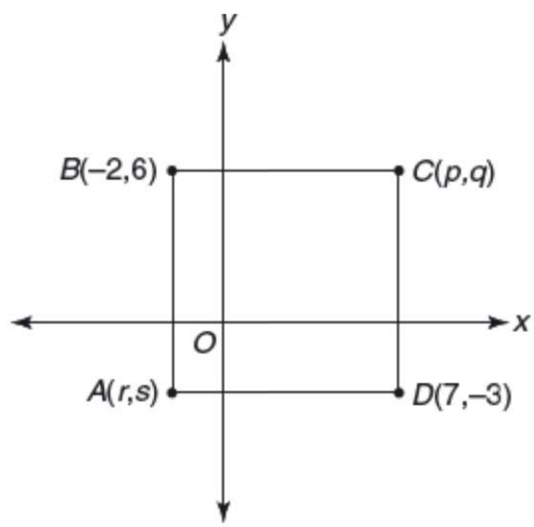
(B) $-\frac{2}{3}$

(C) $\frac{2}{5}$

(D) $\frac{2}{3}$

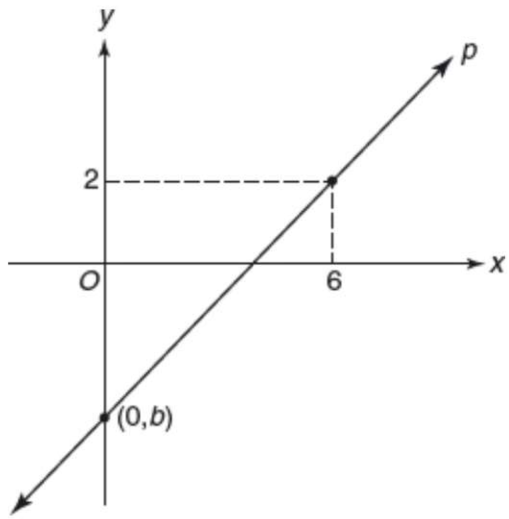
14. Which of the following is an equation of the line that is parallel to the line $y - 4x = 0$ and has the same y-intercept as the line $y + 3 = x + 1$?

(A) $y = 4x - 2$
(B) $y = 4x + 1$
(C) $y = -\frac{1}{4}x + 1$
(D) $y = -\frac{1}{4}x - 2$



15. Which of the following is an equation of the line that contains diagonal $\overline{AC}$ of square $ABCD$ shown in the accompanying figure?

(A) $y = 2x + 1$
(B) $y = -x + 1$
(C) $y = \frac{1}{2}x - 2$
(D) $y = x - 1$



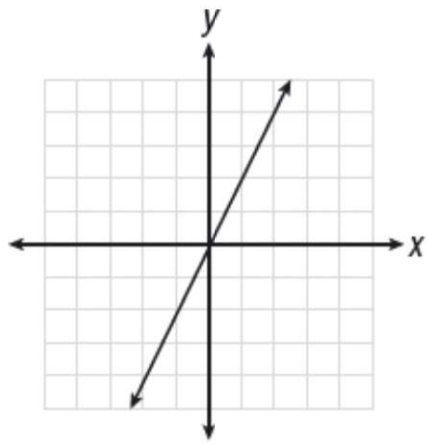
Note: Figure not drawn to scale.

16. If the slope of line p shown in the figure above is $\frac{3}{2}$, what is the value of b?

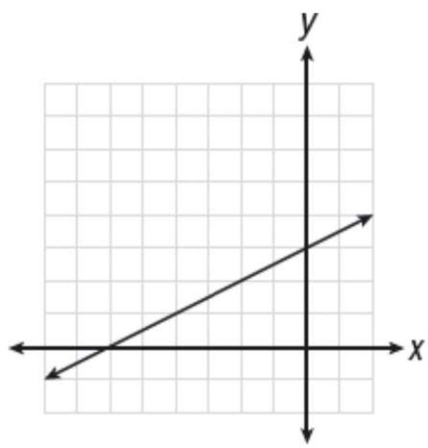
(A) −8
(B) −7
(C) −5
(D) −3

17. Which of the following graphs shows a line where each value of y is three more than half of x?

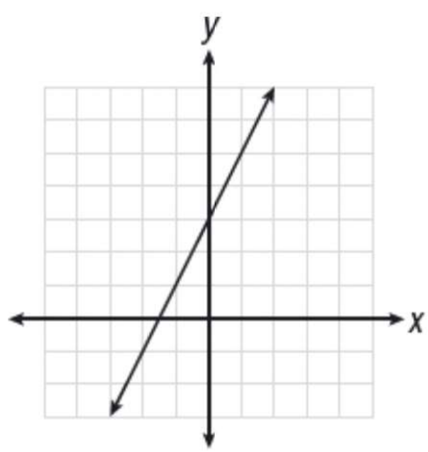
(A) Graph (1)
(B) Graph (2)
(C) Graph (3)
(D) Graph (4)



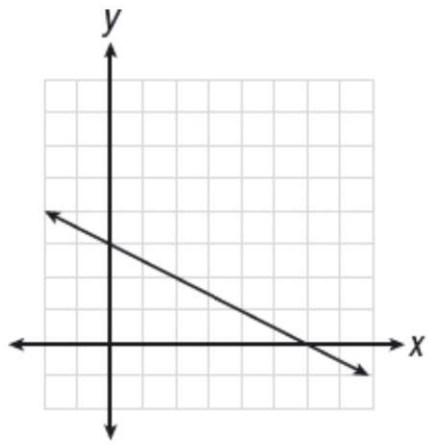
(1)



(2)



(3)



(4)

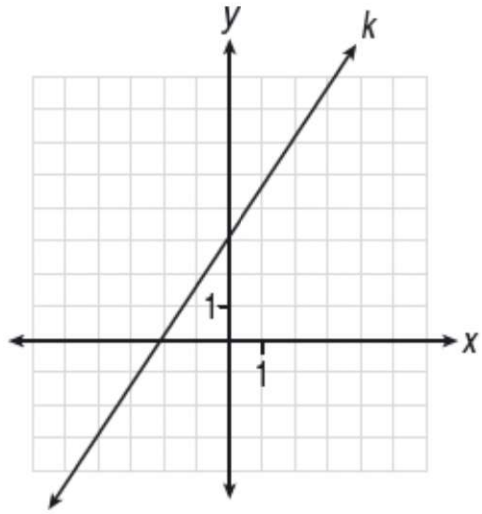
Number of Hours, *h*	Dollars Earned, *d*
8	\$70.00
15	\$113.75
19	\$138.75
30	\$207.50

18. The table above represents the number of hours a student worked and the amount of money the student earned. Which equation represents the number of dollars, d, earned in terms of the number of hours, h, worked

(A) $d = 6.25h$
(B) $d = 6.25h + 20$
(C) $d = 5.25h + 28$
(D) $d = 7h + 8.75$

19. The lines $y = ax + b$ and $y = bx + a$ are graphed in the xy-plane. If a and b are non-zero constants and $a + b = 0$, which statement must be true?

(A) The lines are parallel.
(B) The lines intersect at right angles.
(C) The lines have the same x-intercept.
(D) The lines have the same y-intercept.



20. Which of the following is an equation of the line in the xy-plane that is perpendicular to line k in the figure above?

(A) $y = 3\left(1 - \frac{1}{2}x\right)$

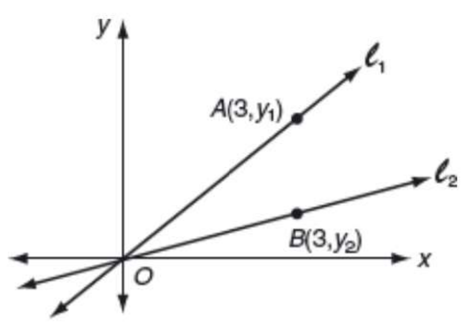
(B) $\frac{x}{y} = \frac{2}{3}$

(C) $3y + 2x = 4$

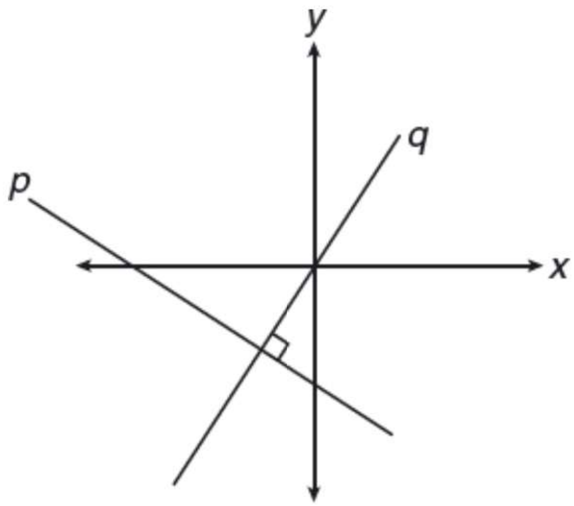
(D) $2y + 3x = -6$

Grid-In

1. A line with a slope of $\frac{3}{14}$ passes through points $(7, 3k)$ and $(0, k)$. What is the value of k?

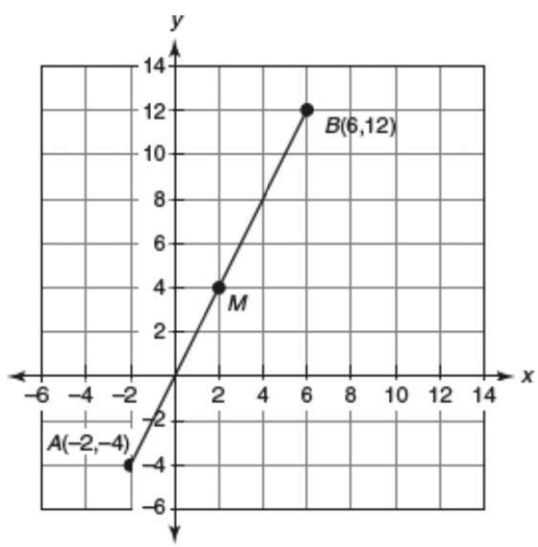


2. In the figure above, the slope of line ℓ_1 is $\frac{5}{6}$ and the slope of line ℓ_2 is $\frac{1}{3}$. What is the distance from point A to point B?



Note: Figure not drawn to scale.

3. In the figure above, lines p and q are perpendicular. Line q passes through the origin and intersects line p at (–1, –2). If (–20, k) is a point on line p, what is the value of k?



4. In the accompanying figure, what is the y-coordinate of the point at which the line that is perpendicular to $\overline{AB}$ (not shown) at point M crosses the y-axis?

5. A line in the xy-plane contains the points $A(c, 40)$ and $B(5, 2c)$. If the line also contains the origin, what is a possible value of c?

LESSON 3-11 GRAPHING LINEAR SYSTEMS

OVERVIEW

If two lines intersect in the *xy*–plane, then their point of intersection represents the solution to the system of equations graphed. When a linear inequality is graphed in the *xy*–plane, the set of all points that satisfy the inequality lie on one side of the line called a **half-plane**. When a system of linear inequalities is graphed in the *xy*–plane, the solution set is the region over which the solution half-planes of the individual inequalities overlap.

SYSTEMS OF LINEAR EQUATIONS

If a system of two linear equations is graphed in the *xy*-plane, then there are three possibilities to consider:

(1) The Lines Intersect

The coordinates of the point of intersection represent the solution to the linear system. The linear system $y = 2x + 1$ and $y = -x + 4$ consists of two lines with different slopes and, as a result, intersect at a single point that satisfies both equations. See Figure 3.9.

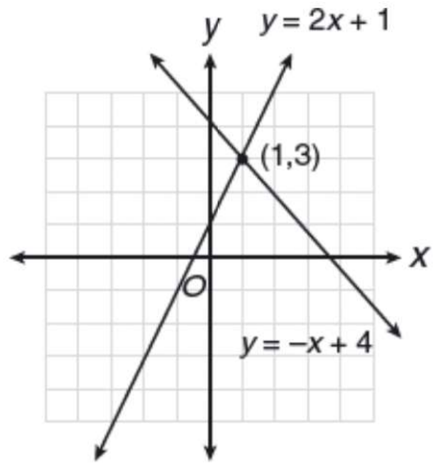


Figure 3.9 Lines with different slopes

(2) The Lines Do *Not* Intersect

If the lines are different and parallel, as in the linear system $y = 2x + 4$ and $y = 2x - 1$ shown in Figure 3.10, the system has no solution since there is no point common to both lines.

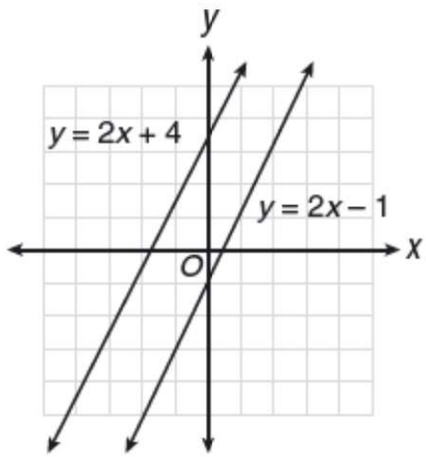


Figure 3.10 Lines with the same slope and different *y*-intercepts

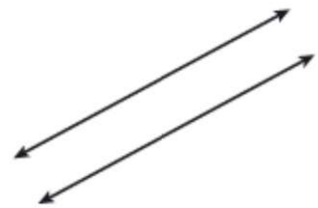
(3) The Lines Coincide

The linear system $x - y = 3$ and $2x - 2y = 6$ consists of two lines that coincide as one equation is a multiple of the other. The lines have the same slope and also have the same *y*-intercepts. Because each point on the lines is a solution, there are infinitely many solutions.

The three possibilities are summarized in Table 3.2.

> **TIP**
>
> **To determine the number of solutions of a linear system, compare the equations of the two lines in $y = mx + b$ slope-intercept form.**

Table 3.2 Possible Solutions to a Linear System

One Solution	No Solution	Infinitely Many Solutions
 Intersecting lines: lines have different slopes.	 Parallel lines: lines have the same slope and different *y*-intercepts.	 Lines coincide: lines have the same slope *and* the same *y*-intercepts. The two equations represent the same line.

➥ Example

$$\frac{2}{3}x - \frac{1}{2}y = 7$$
$$kx - 6y = 4$$

In the system of linear equations above, k is a constant. If the system has no solution, what is the value of k?

Solution

Since the system has no solution, the graphs of the equations in the xy-plane are parallel lines so the lines have the same slope. Write each equation in $y = mx + b$ slope-intercept form and then compare their slopes.

- If $\frac{2}{3}x - \frac{1}{2}y = 7$, then $-\frac{1}{2}y = -\frac{2}{3}x + 7$ so $y = \frac{4}{3}x - 14$.
- If $kx - 6y = 4$, then $-6y = -kx + 4$ so $y = \frac{k}{6}x - \frac{2}{3}$.
- When an equation of a line is written in $y = mx + b$ form, the coefficient of the x-term represents the slope of the line. Since parallel lines have equal slopes, $\frac{4}{3} = \frac{k}{6}$ so $3k = 24$ and $k = \frac{24}{3} = 8$.

The value of k is **8**.

GRAPHING A LINEAR INEQUALITY

To graph a linear inequality such as $y + 2x < 6$,

- Replace the inequality relation with an equal sign (=) and graph the boundary line. Draw a solid line for a $\geq$ or $\leq$ inequality relation and a broken line if the inequality relation is > or <. See Figure 3.11.

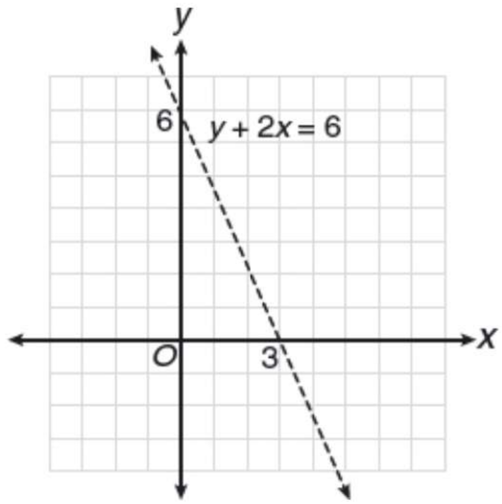


Figure 3.11 Graphing the boundary line of $y + 2x < 6$

- If after solving for y in terms of x, the inequality relation is $\geq$ or >, then the solution region lies *above* the boundary line; if the inequality relation is $\leq$ or <, the solution region lies *below* the boundary line. Since $y + 2x < 6$ becomes $y < -2x + 6$ and the inequality is "less than," the solution region lies *below* the line, as shown in Figure 3.12.

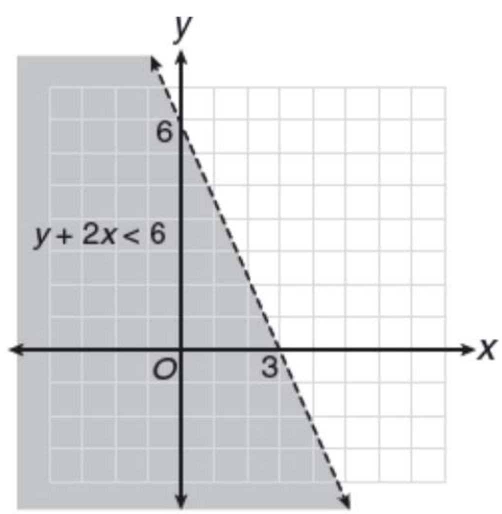


Figure 3.12 Shaded region represents all points that satisfy $y < -2x + 6$

SYSTEMS OF LINEAR INEQUALITIES

The solution set for a system of two linear inequalities is the region over which the solution sets for the two linear inequalities overlap.

Consider the system:

$$y \leq x + 1$$
$$y > -\frac{1}{2}x + 4$$

- The solution set for $y \leq x + 1$ consists of the set of points that lie *below* ($\leq$) or on the boundary line $y = x + 1$. See Figure 3.13.

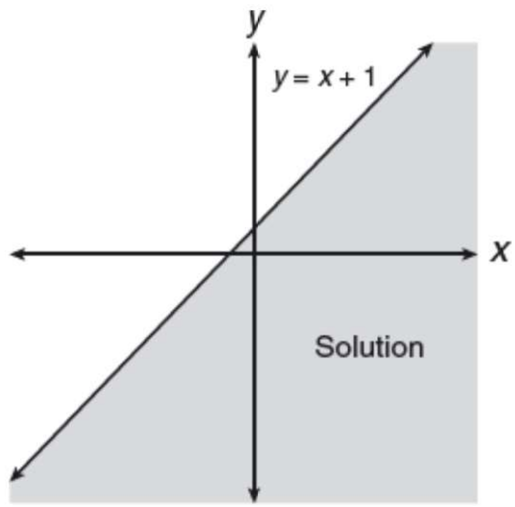


Figure 3.13 Graphing boundary line of $y \leq x + 1$

- The solution set for $y > -\frac{1}{2}x + 4$ consists of the set of all points that lie *above* (>) the boundary line $y = -\frac{1}{2}x + 4$ as shown in Figure 3.14.

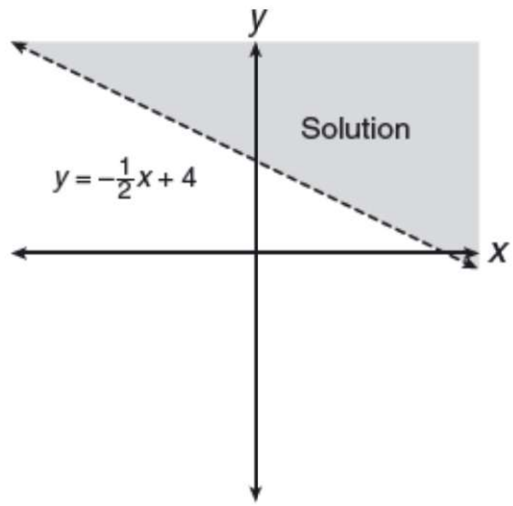


Figure 3.14 Graphing boundary line of $y > -\frac{1}{2}x + 4$

- The solution set for the system lies in the region (labeled I) where the two solution sets overlap, as illustrated in Figure 3.15.

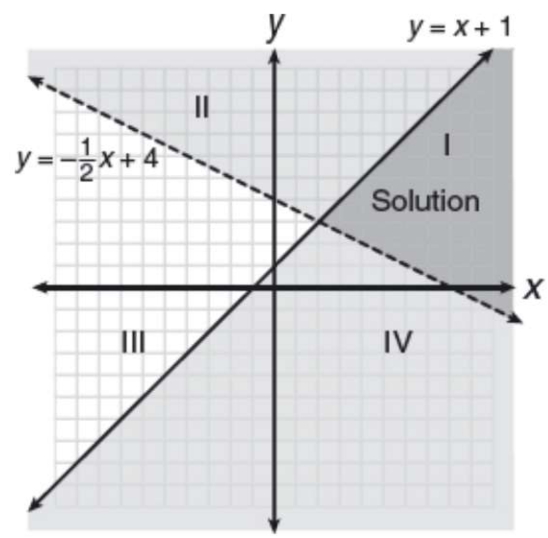


Figure 3.15 Graphing solution of $y \leq x + 1$ and $y > -\frac{1}{2}x + 4$

You can also determine which of the four regions above represents the solution set to the system of linear inequalities by picking a point in each region and testing whether it makes both inequalities true at the same time. For example, test the point (6, 2), which lies in region I:

Test (6, 2): $y \le x + 1$

$2 \le 6 + 1$

$2 \le 7$ *True*!

Test (6, 2): $y \ge -x + 4$

$2 \ge -(6) + 4$

$2 \ge -2$ *True*!

Hence, the bounded region that contains (6, 2) represents the solution set.

LESSON 3-11 TUNE-UP EXERCISES

Multiple-Choice

$$4x + 6y = 12$$
$$y = 8 - kx$$

1. For what value of k does the system of equations above have no solution?

   (A) $-\frac{3}{2}$

   (B) 0

   (C) $\frac{2}{3}$

   (D) 4

2. Sara correctly solves a system of two linear equations and finds that the system has no solution. If one of the two equations is $\frac{y}{6} - \frac{x}{4} = 1$, which could be the other equation in this system?

   (A) $y = \frac{2}{3}x + 12$

   (B) $y = \frac{3}{2}x$

   (C) $y = -\frac{3}{2}x$

   (D) $y = \frac{3}{2}x + 6$

3. Ben correctly solves a system of two linear equations and finds that the system has an infinite number of solutions. If one of the two equations is $3(x + y) = 6 - x$, which could be the other equation in this system?

(A) $y = \frac{3}{4}x + 2$

(B) $y = -\frac{4}{3}x$

(C) $y = -\frac{4}{3}x + 2$

(D) $y = -\frac{4}{3}x + 6$

4. The graph of the inequality $y \leq 2x$ will include all of the points in which quadrant?
   (A) Quadrant I
   (B) Quadrant II
   (C) Quadrant III
   (D) Quadrant IV

$$\frac{1}{2}x - \frac{5}{6}y = 5$$
$$-2x + ky = 3$$

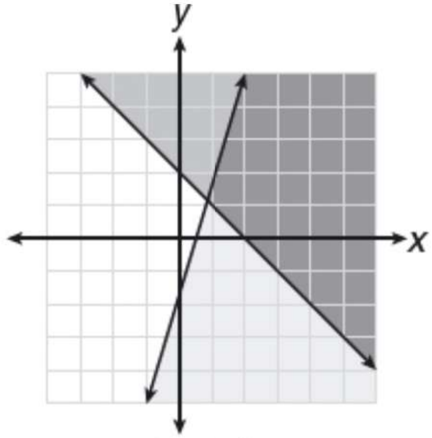
5. In the system of linear equations above, k is a constant. If the system has no solution, what is the value of k?
   (A) $\frac{5}{3}$
   (B) $\frac{5}{2}$
   (C) $\frac{10}{3}$
   (D) $\frac{15}{2}$

6. The graph of a line in the xy-plane has slope $\frac{1}{2}$ and contains the point (0, 7). The graph of a second line passes through the points (0, 0) and (–1, 3). If the two lines intersect at the point (r, s), what is the value of $r + s$?
   (A) –3
   (B) –2
   (C) 2
   (D) 4

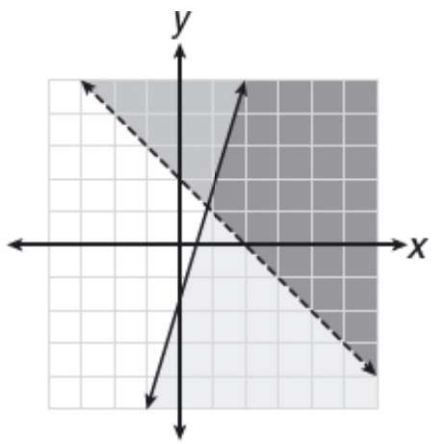
$$y + x > 2$$

$$y \leq 3x - 2$$

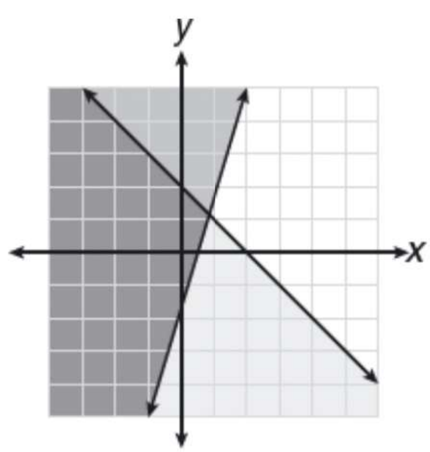
7. Which graph shows the solution of the set of inequalities above?



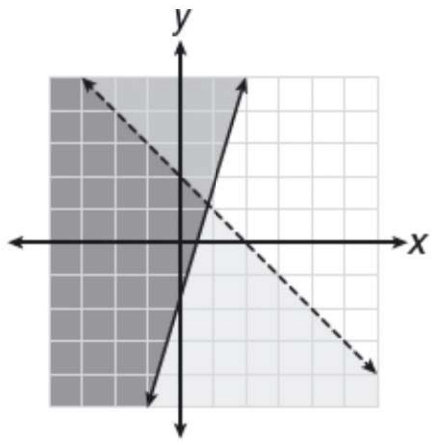
Graph A



Graph B

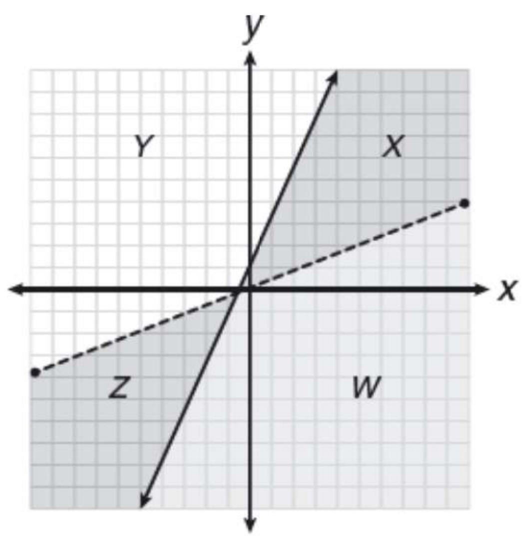


Graph C



Graph D

(A) Graph A
(B) Graph B
(C) Graph C
(D) Graph D



$$2y < x$$

$$y \geq 3x + 1$$

8. A system of inequalities and a graph are shown above. Which region or regions of the graph could represent the set of all ordered pairs that satisfy the system?

(A) Region X
(B) Regions X and Z
(C) Regions X, Y, and W

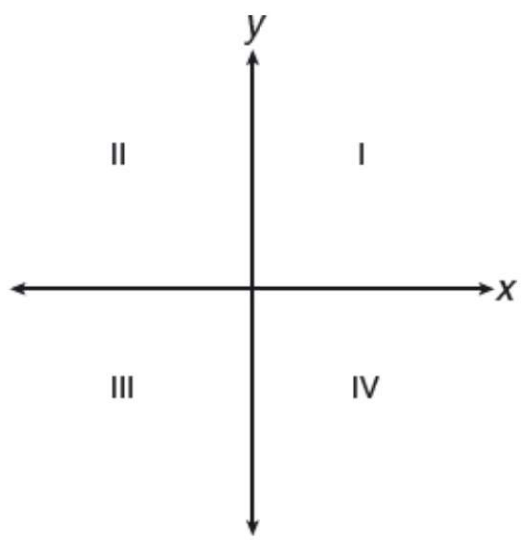
(D) Region Z

$$3x + 5 = 2y$$

$$\frac{x}{3} + \frac{y}{2} = \frac{2}{3}$$

9. For the system of equations above, which of the following statements is true?

(A) The system has no solution.
(B) The graphs of the equations in the xy-plane intersect at right angles.
(C) The graphs of the equations in the xy-plane intersect but *not* at right angles.
(D) The system has infinitely many solutions.



10. If the system of inequalities $y < 2x + 4$ and $y \geq -x + 1$ is graphed in the xy-plane above which quadrant does not contain any solutions to the system?

(A) Quadrant I
(B) Quadrant II
(C) Quadrant III
(D) Quadrant IV

Grid-In

$$6x + py = 21$$
$$qx + 5y = 7$$

1. If the above system of equations has infinitely many solutions, what is the value of $\frac{p}{q}$?

$$(k - 1)x + \frac{1}{3}y = 4$$
$$k(x + 2y) = 7$$

2. In the system of linear equations above, k is a constant. If the system has no solution, what is the value of k?

$$\frac{1}{3}r + 4s = 1$$
$$kr + 6s = -5$$

3. In the system of equations above, k and s are nonzero constants. If the system has no solutions, what is the value of k?

4. The graph of a line in the xy-plane passes through the points (5, –5) and (1, 3). The graph of a second line has a slope of 6 and passes through the point (–1, 15). If the two lines intersect at (p, q), what is the value of $p + q$?

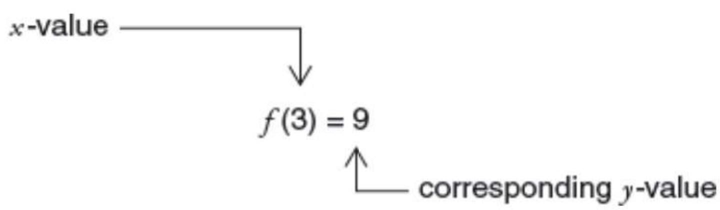
LESSON 3-12 WORKING WITH FUNCTIONS

OVERVIEW

A **function** is a *rule* that tells how to pair each member of one set (say the x-values) with exactly one member of a second set (say the y-values). The rule is typically expressed as either a "$y = \ldots$" type of equation, a graph, or a table. The set of all possible x-values (input) is the **domain** and the resulting set of y-values (output) is the **range**.

EQUATIONS AS FUNCTIONS

A function is usually named by a lowercase letter, such as f or g. The equation $y = 2x + 3$ describes a function, since it gives a rule for pairing any given x-value with one particular y-value: input any number x, multiply it by 2, add 3, and name the result y. When $x = 3$, $y = 2(3) + 3 = 9$. If this function is called f, then the ordered pair (3, 9) belongs to function f. This fact can be abbreviated by writing $f(3) = 9$, which is read as "f of three equals nine":



MATH REFERENCE FACT

The shorthand notation $f(x)$ represents the value of function f when x has the value inside the parentheses.

Here are some examples of evaluating functions:

- If $h(x) = \frac{x}{x^3 + 1}$, then to find $h(2)$, replace x with 2:

$$h(2) = \frac{2}{(2)^3 + 1} = \frac{2}{8+1} = \frac{2}{9}$$

- If $g(x) = \frac{1}{2}x + 2$, then to find $4g(x) + 1$, multiply the equation that defines function g by 4 and then add 1:

$$4g(x) + 1 = 4\overbrace{\left[\frac{1}{2}x + 2\right]}^{g(x)} + 1 = (2x + 8) + 1 = 2x + 9$$

- If $f(x) = x^2 + x$, then to find $f(n - 1)$, replace each x with $n - 1$:

$$\begin{aligned} f(n-1) &= (n-1)^2 + (n-1) \\ &= (n-1)(n-1) + (n-1) \\ &= (n^2 - 2n + 1) + (n-1) \\ &= n^2 - n \end{aligned}$$

➡ Example

If $k(x) = \frac{2x - p}{5}$ and $k(7) = 3$, what is the value of p?

Solution

Since $k(7) = 3$, replace x with 7 and set the result equal to 3:

$$\begin{aligned} k(7) &= \frac{2(7) - p}{5} \\ 3 &= \frac{14 - p}{5} \\ 15 &= 14 - p \\ p &= 14 - 15 \\ p &= -1 \end{aligned}$$

➡ Example

Function f is defined by $f(x) = 2x + 1$. If $2f(m) = 30$, what is the value of $f(2m)$?

Solution

- If $2f(m) = 30$, then $f(m) = \frac{30}{2} = 15$.
- Since $f(m) = 2m + 1 = 15$, $2m = 14$ and $m = 7$.
- Hence, $f(2m) = f(14) = 2(14) + 1 = 29$.

Grid-in **29**

➡ Example

Function f is defined by $f(x) = \frac{3}{2}x + c$. If $f(6) = 1$, what is the value of $f(c)$?

(A) –20
(B) –8
(C) 4
(D) 12

Solution

- Find the value of c:

$$f(6) = \frac{3}{\cancel{2}} \times \overset{3}{\cancel{(6)}} + c = 1$$
$$9 + c = 1$$
$$c = -8$$

- Since $c = -8$, $f(x) = \frac{3}{2}x - 8$.
- Evaluate $f(-8)$:

$$f(x) = \frac{3}{2}x - 8$$
$$f(-8) = \frac{3}{2}(-8) - 8$$
$$= 3(-4) - 8$$
$$= -20$$

The correct choice is **(A)**.

➥ Example

If $g(x) = 3x - 1$ and $f(x) = 4g(x) + 3$, what is $f(2)$?

Solution

- Substitute 2 for x in $f(x) = 4g(x) + 3$:

$$f(2) = 4g(2) + 3$$

- Use $g(x) = 3x - 1$ to find the value of $g(2)$:

$$g(2) = 3(2) - 1 = 5$$

- Hence, $f(2) = 4(5) + 3 = 23$.

➥ Example

If $f(x) = x^2 - 2x$, what is $f(2x + 1)$?

(A) $4x + 2$
(B) $4x^2 - 1$
(C) $4x^2 + 1$
(D) $4x^2 - 4x - 1$

Solution

Since $f(x) = x^2 - 2x$,

$$\begin{aligned} f(2x + 1) &= (2x + 1)^2 - 2(2x + 1) \\ &= (4x^2 + 4x + 1) - 4x - 2 \\ &= 4x^2 - 1 \end{aligned}$$

The correct choice is **(B)**.

DETERMINING THE DOMAIN AND RANGE

Unless otherwise indicated, the *domain* of function f is the largest possible set of real numbers x for which $f(x)$ is a real number. There are two key rules to follow when finding the domain of a function:

1. Do ***not*** divide by zero. Exclude from the domain of a function any value of x that results in division by 0. If $f(x) = \frac{x+2}{x-1}$, then $f(1) = \frac{1+2}{1-1} = \frac{3}{0}$. Since division by 0 is not allowed, x cannot be equal to 1. The domain of function f is the set of all real numbers *except* 1.
2. Do ***not*** take the square root of a negative number. Since the square root of a negative number is not a real number, the quantity underneath a square root sign must always evaluate to a number that is greater than or equal to 0. If $f(x) = \sqrt{x-3}$, then x must be at least 3, since any lesser value of x will result in the square root of a negative number. For example, $f(1) = \sqrt{1-3} = \sqrt{-2}$, but $\sqrt{-2}$ is not a real number. Thus, the domain of function f is limited to the set of all real numbers greater than or equal to 3.

The *range* of the function $y = f(x)$ is the set of all values that y can have as x takes on each of its possible values. For example, if function f is defined by $f(x) = 1 + \sqrt{x}$, then the smallest possible function value is $f(0) = 1 + \sqrt{0} = 1$.

As x increases without bound, so does $1 + \sqrt{x}$. Therefore, the range of f is the set of all real numbers greater than or equal to 1.

COMPOSITE FUNCTION NOTATION

The output of one function can be used as the input of a second function. The notation $f(g(2))$ means that the output of function g when $x = 2$ is used as the input for function f. If $f(x) = 3x - 1$ and $g(x) = x^2$, then

$$g(2) = 2^2 = 4 \text{ so } f(g(2)) = f(4) = 3(4) - 1 = 11$$

In general, $f(g(x))$ and $g(f(x))$ do *not* necessarily represent the same value. For example, using the same definitions for functions f and g,

$$f(2) = 3(2) - 1 = 5 \text{ so } g(f(2)) = g(5) = 5^5 = 25$$

➡ Example

x	0	1	4	5
$f(x)$	–2	4	0	2

x	2	1	3	–4
$g(x)$	0	2	1	5

Some values of functions f and g are given by the tables above. What is the value of $f(g(3))$?

(A) –2
(B) 0
(C) 2
(D) 4

Solution

If $x = 3$, $g(3) = 1$. Then

$$f(g(3)) = f(1) = 4$$

The correct choice is **(D)**.

REPRESENTING A SEQUENCE OF NUMBERS

An ordered sequence of numbers can be represented using function notation. For example, if the first term of an ordered sequence of integers is 4, then $f(1) = 4$. If the nth term of this sequence is given by the function $f(n) = f(n - 1) + n$, then successive terms of the sequence after the first can be obtained by replacing n with 2, 3, 4, and so forth. For example,

- To find the second term of this sequence, set $n = 2$:

$$f(2) = f(2 - 1) + 2 \text{ so } f(2) = f(1) + 2$$

  Since it is given $f(1) = 4$, $f(2) = 4 + 2 = 6$.
- To find the third term of this sequence, set $n = 3$:

$$f(3) = f(3 - 1) + 3 \text{ so } f(3) = f(2) + 3$$

  From the previous step, we know that $f(2) = 6$, so $f(3) = 6 + 3 = 9$.

Hence, the first three terms of the sequence defined by function f are 4, 6, and 9.

> **MATH REFERENCE FACT**
>
> A function may be defined *explicitly* or *recursively*. A function defined in the usual way as a "$y = \ldots$" type of equation is defined explicitly. A function such as $f(n) = f(n - 1) + n$ where $f(1) = 4$ is defined recursively since each of its values after the first is calculated using function values that come before it.

GRAPHS AS FUNCTIONS

Since a graph is a set of ordered pairs located on a coordinate grid, a function may take the form of a graph. If Figure 3.16 shows the complete graph for function g, then you can tell the following from the graph:

- The *domain* of g is $-6 \le x \le 5$, since the greatest set of x-values over which the graph extends *horizontally* is from $x = -6$ to $x = 5$, inclusive.
- The *range* of g is $0 \le y \le 8$, because the greatest set of y-values over which the graph extends *vertically* is from $y = 0$ to $y = 8$, inclusive.

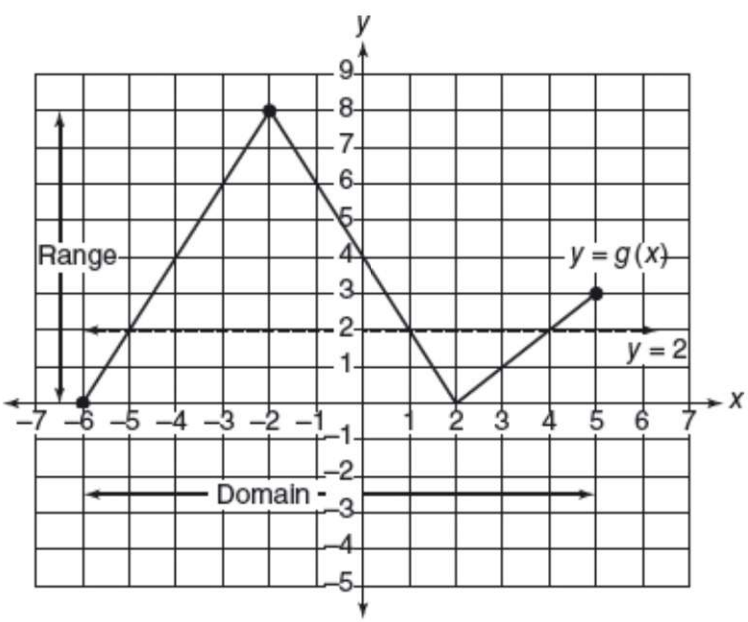


Figure 3.16 Domain and range of function g. The line $y = 2$ intersects the graph at points for which $g(x) = 2$

You can also read specific function values from the graph of function g in Figure 3.16.

- To find $g(-1)$, determine the y-coordinate of the point on the graph whose x-coordinate is -1. Since the graph contains $(-1, 6)$, $g(-1) = 6$.
- To find the values of t such that $g(t) = 2$, find all points on the graph whose y-coordinates are 2 by drawing a horizontal line through $(0, 2)$. Since the graph passes through $(-5, 2)$, $(1, 2)$, and $(4, 2)$, $g(-5) = 2$, $g(1) = 2$, and $g(4) = 2$. Hence, the possible values of t are -5, 1, and 4.

TIP

Given the graph of function f, you can find those values of t that make $f(t) = k$ by drawing a horizontal line through k on the y-axis. The x-coordinates of the points at which the horizontal line intersects the graph, if any, represent the possible values of t.

FINDING THE ZEROS OF A FUNCTION

The zeros of a function f are those values of x, if any, for which $f(x) = 0$. You can determine the zeros of a function from its graph by locating the points at which the graph intersects the x-axis. At each of these points, the y-coordinate is 0, so $f(x) = 0$.

MATH REFERENCE FACT

The x-intercepts of the graph of function f, if any, correspond to those values of x for which $f(x) = 0$.

➥ Example

Referring to Figure 3.16, which could be the value of s when $g(s) = 0$?

I. -6 II. 2 III. 4

(A) I only
(B) II only
(C) III only
(D) I and II only

Solution

Since the graph of function g has x-intercepts at $x = -6$ and $x = 2$, $g(-6) = 0$ and $g(2) = 0$. Because s can be equal to either -6 or 2, Roman numeral choices I and II are correct. The correct choice is **(D)**.

➡ Example

The zeros of the function $f(x) = 25 - (x + 2)^2$ are

(A) –2 and 5
(B) –3 and 7
(C) –5 and 2
(D) –7 and 3

Solution

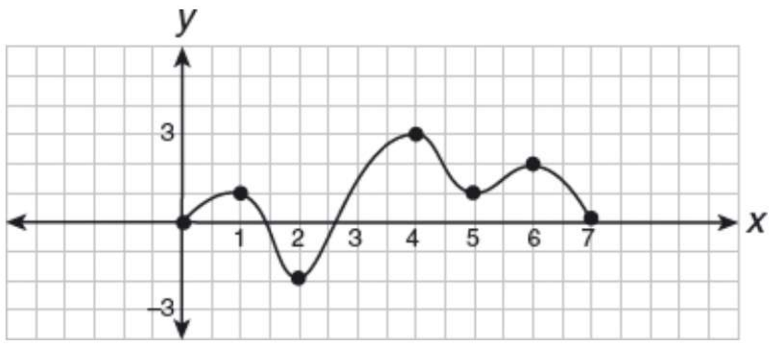
The zeros of a function $f(x)$ are those x-values that make the function evaluate to 0. Thus, to find the zeros of function f, set $f(x)$ equal to 0:

$$25 - (x + 2)^2 = 0$$

Since $(-7 + 2)^2 = (3 + 2)^2 = 25$, the equation will be true when $x = -7$ or $x = 3$.

The correct choice is **(D)**.

➡ Example



The graph above shows the function $y = f(x)$ over the interval $0 \leq x \leq 7$. What is the value of $f(f(6))$?

(A) -2
(B) 0
(C) 1
(D) 2

Solution

Reading from the graph, $f(6) = 2$ so $f(f(6)) = f(2) = -2$.
The correct choice is **(A)**.

LESSON 3-12 TUNE-UP EXERCISES

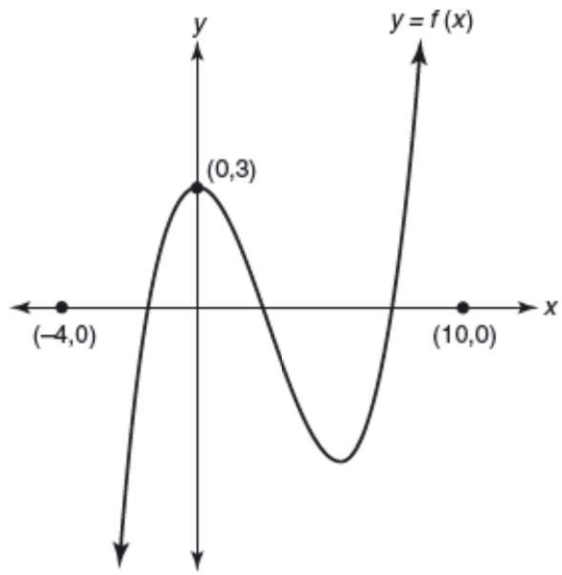
Multiple-Choice

1. If the function f is defined by $f(x) = 3x + 2$, and if $f(a) = 17$, what is the value of a?

   (A) 5
   (B) 9
   (C) 10
   (D) 11

2. A function f is defined such that $f(1) = 2$, $f(2) = 5$, and $f(n) = f(n - 1) - f(n - 2)$ for all integer values of n greater than 2. What is the value of $f(4)$?

   (A) −8
   (B) −2
   (C) 2
   (D) 8



3. The graph of $y = f(x)$ is shown above. If $-4 \leq x \leq 10$, for how many values of x does $f(x) = 2$?

   (A) None
   (B) One

(C) Two
(D) Three

4. If function f is defined by $f(x) = 5x + 3$, then which expression represents $2f(x) - 3$?

(A) $10x - 3$
(B) $10x + 3$
(C) $10x$
(D) 3

5. If the function k is defined by $k(h) = (h + 1)^2$, then $k(x - 2) =$

(A) $x^2 - x$
(B) $x^2 - 2x$
(C) $x^2 - 2x + 1$
(D) $x^2 + 2x - 1$

x	1	2	3	4	5
$f(x)$	3	4	5	6	7

x	3	4	5	6	7
$g(x)$	4	6	8	10	12

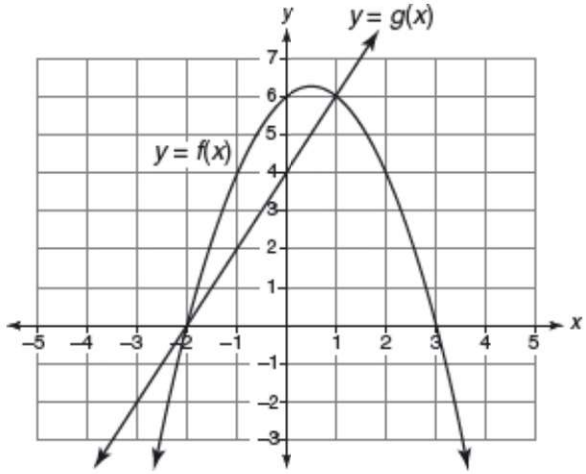
6. The accompanying tables define functions f and g. What is $g(f(3))$?

(A) 4
(B) 6
(C) 8
(D) 10

7. In 2014, the United States Postal Service charged \$0.48 to mail a first-class letter weighing up to 1 oz. and \$0.21 for each additional ounce. Based on these rates, which function would determine the cost, in dollars, $c(z)$, of mailing a first-class letter weighing z ounces where z is an integer greater than 1?

(A) $c(z) = 0.48z + 0.21$
(B) $c(z) = 0.21z + 0.48$
(C) $c(z) = 0.48(z - 1) + 0.21$

(D) $c(z) = 0.21(z - 1) + 0.48$



8. Based on the graphs of functions f and g shown in the accompanying figure, for which values of x between -3 and 3 is $f(x) \geq g(x)$?

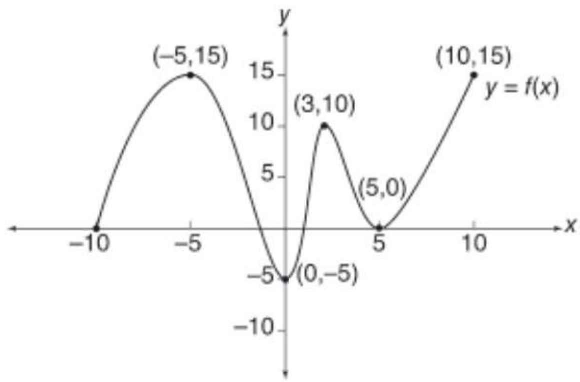
I. $-2 \leq x \leq 0$
II. $0 \leq x \leq 1$
III. $1 \leq x \leq 3$

(A) I only
(B) II only
(C) III only
(D) I and II

- $f(2n) = 4f(n)$ for all integers n
- $f(3) = 9$

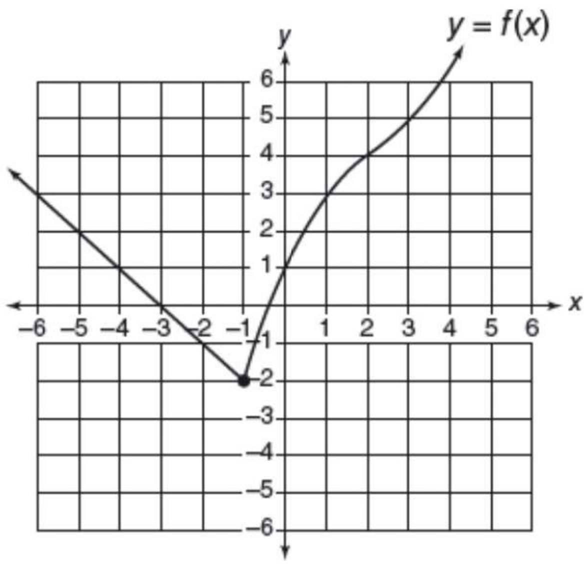
9. If function f satisfies the above two conditions for all positive integers n, which equation could represent function f?

(A) $f(n) = 9$
(B) $f(n) = n^2$
(C) $f(n) = 3n$
(D) $f(n) = 2n + 3$



10. If in the accompanying figure (p, q) lies on the graph of $y = f(x)$ and $0 \leq p \leq 5$, which of the following represents the set of corresponding values of q?

(A) $-5 \leq q \leq 15$
(B) $-5 \leq q \leq 10$
(C) $-5 \leq q \leq 5$
(D) $5 \leq q \leq 10$



11. The accompanying figure shows the graph of $y = f(x)$. If function g is defined by $g(x) = f(x + 4)$, then $g(-1)$ could be

(A) −2
(B) 3
(C) 4
(D) 5

x	$f(x)$	$g(x)$
1	2	3
2	4	5
3	5	1
4	3	2
5	1	4

Questions 12–13 refer to the accompanying table, which gives the values of functions f and g for integer values of x from 1 to 5, inclusive.

12. According to the table, if $f(5) = p$, what is the value of $g(p)$?

    (A) 1
    (B) 2
    (C) 3
    (D) 4

13. Function h is defined by $h(x) = 2f(x) - 1$, where function f is defined in the accompanying table. What is the value of $g(k)$ when $h(k) = 5$?

    (A) 1
    (B) 2
    (C) 3
    (D) 4

x	0	1	4	5
$f(x)$	–2	5	0	2

x	0	2	3	–4
$g(x)$	2	–1	1	5

14. Some values of functions f and g are given by the tables above. What is the value of $g(f(5))$?

    (A) –1
    (B) 1
    (C) 2
    (D) 5

15. In 2012, a retail chain of fast food restaurants had 68 restaurants in California and started to expand nationally by adding 9 new restaurants each year thereafter. At this rate, which of the following functions f

represent the number of restaurants there will be in this retail chain n years after 2012 assuming none of these restaurants close?

(A) $f(n) = 2{,}012 + 9n$
(B) $f(n) = 9 + 68n$
(C) $f(n) = 68 + 9(n - 2{,}012)$
(D) $f(n) = 68 + 9n$

16. According to market research, the number of magazine subscriptions that can be sold can be estimated using the function

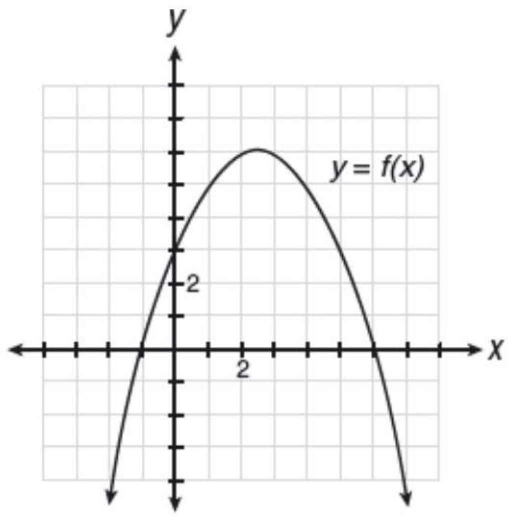
$$n(p) = \frac{5{,}000}{4p - k},$$

where n is the number of thousands of subscriptions sold, p is the price in dollars for each individual subscription, and k is some constant. If 250,000 subscriptions were sold at \$15 for each subscription, how many subscriptions could be sold if the price were set at \$20 for each subscription?

(A) 50,000
(B) 75,000
(C) 100,000
(D) 125,000

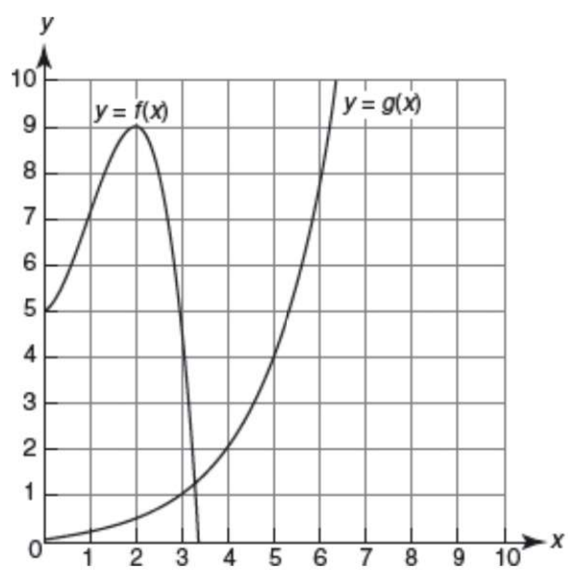
Grid-In

1. Let h be the function defined by $h(x) = x + 4^x$. What is the value of $h\left(-\frac{1}{2}\right)$?



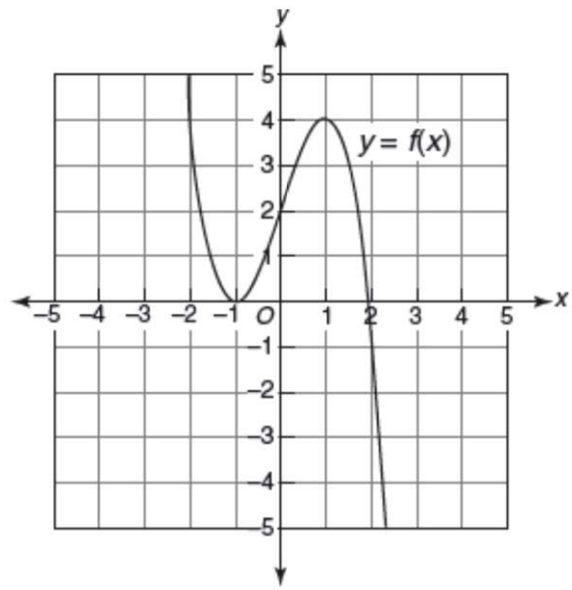
2. The above figure shows the graph of $y = f(x)$ where c is a nonzero constant. If $f(w + 1.7) = 0$ and $w > 0$, what is a possible value of w?

3. Let the function f be defined by $f(x) = x^2 + 12$. If n is a positive number such that $f(3n) = 3f(n)$, what is the value of n?



4. Let the functions f and g be defined by the graphs in the accompanying diagram. What is the value of $f(g(3))$?

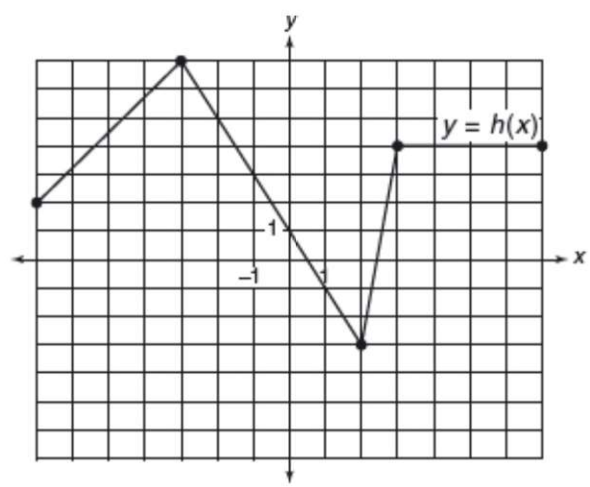
Questions 5 and 6 Let the function f be defined by the graph below.



5. What is the integer value of $2f(-1) + 3f(1)$?

6. If n represents the number of different values of x for which $f(x) = 2$ and m represents the number of different values of x for which $f(x) = 4$, what is the value of mn?

7. Let g be the function defined by $g(x) = x - 1$. If $\frac{1}{2}g(c) = 4$, what is the value of $g(2c)$?



8. The figure above shows the graph of function h. If function f is defined by $f(x) = h(2x) + 1$, what is the value of $f(-1)$?

Problem Solving and Data Analysis 4

The beginning lessons of this chapter focus on mathematical reasoning involving algebraic representations, percent, and ratios. Because of their importance, linear and exponential functions receive special attention. The chapter also covers analyzing and summarizing data using graphs, scatter plots, tables, and basic statistical measures. "Problem Solving and Data Analysis" represents the second of the four major mathematics content groups tested by the redesigned SAT.

LESSONS IN THIS CHAPTER

Lesson 4-1 Working with Percent
Lesson 4-2 Ratio and Variation
Lesson 4-3 Rate Problems
Lesson 4-4 Converting Units of Measurement
Lesson 4-5 Linear and Exponential Functions
Lesson 4-6 Graphs and Tables
Lesson 4-7 Scatterplots and Sampling
Lesson 4-8 Summarizing Data Using Statistics

LESSON 4-1 WORKING WITH PERCENT

OVERVIEW

Percent means parts out of 100. Simple algebraic equations can be used to help solve a variety of problems involving percent.

THE THREE TYPES OF PERCENT PROBLEMS

You can solve each of the three basic types of percent problems by writing and solving an equation.

- **Type 1:** Finding a percent of a given number.

➡ Example

$$\underbrace{\text{What}}_{\downarrow\; n}\ \underbrace{\text{is}}_{\downarrow\; =}\ \underbrace{15\%}_{\downarrow\; 0.15}\ \underbrace{\text{of}}_{\downarrow\; \times}\ \underbrace{80?}_{\downarrow\; 80}$$

$$n = 0.15 \times 80$$

$$= 12$$

15% of 80 is 12.

- **Type 2:** Finding a number when a percent of it is given.

➡ Example

$$\underbrace{30\%}_{\downarrow\; 0.30}\ \underbrace{\text{of}}_{\downarrow\; \times}\ \underbrace{\text{what number}}_{\downarrow\; n}\ \underbrace{\text{is}}_{\downarrow\; =}\ \underbrace{12?}_{\downarrow\; 12}$$

$$0.30 \times n = 12$$

$$0.30n = 12$$

$$10(0.3n) = 10(12)$$

$$3n = 120$$

$$n = \frac{120}{3}$$

$$= 40$$

30% of 40 is 12.

- **Type 3:** Finding what percent one number is of another.

➥ Example

What percent of 30 is 9?

$$\frac{p}{100} \times 30 = 9$$

$$\frac{p}{100} \times 30 = 9$$

$$\frac{\overset{3}{\cancel{30}}p}{\underset{10}{\cancel{100}}} = 9$$

$$p = \frac{10 \cdot 9}{3} = 30$$

9 is 30% of 30.

Finding an Amount After a Percent Change

If a 20% tip is left on a restaurant bill of $80, then to find the total amount of the bill including the tip, do the following:

STEP 1 Find the amount of the tip: $80 × 0.20 = $16

STEP 2 Add the tip to the bill: $80 + $16 = $96

> **TIME SAVER**
>
> Find the final result of increasing or decreasing a number by a percent in one step by multiplying the number by the *total* percentage:
>
> - If 80 is *increased* by 20%, the total percentage is 100% + 20% = 120%, so the final amount is 80 × 1.20 = 96.
> - If 50 is *decreased* by 30%, the total percentage is 100% − 30% = 70%, so the final amount is 50 × 0.70 = 35.

➥ Example

If the length of a rectangle is increased by 30% and its width is increased by 10%, by what percentage will the area of the rectangle be increased?

(A) 33%
(B) 37%
(C) 40%
(D) 43%

Solution

Pick easy numbers for the length and width of the rectangle. Assume the length and width of the rectangle are each 10 units, so the area of the original rectangle is $10 \times 10 = 100$ square units.

- The total percent increase of the length is $100\% + 30\% = 130\%$. The length of the new rectangle is $10 \times 1.3 = 13$.
- The total percent increase of the width is $100\% + 10\% = 110\%$. The width of the new rectangle is $10 \times 1.1 = 11$.
- The area of the new rectangle is $13 \times 11 = 143$ square units. Compared to the original area of 100 square units, this is a 43% increase.

The correct choice is **(D)**.

Finding an Original Amount after a Percent Change

If you know the number that results after a given number is increased by $P\%$, you can find the original number by dividing the new amount by the total percentage:

$$\text{Original amount} = \frac{\text{New amount after an increase of } P\%}{100\% + P\%}$$

Similarly, if you know the number that results after a given number is decreased by $P\%$, you can find the original number by dividing the new amount by the total percentage:

$$\text{Original amount} = \frac{\text{New amount after a decrease of } P\%}{100\% - P\%}$$

➡ **Example**

A pair of tennis shoes cost $48.60 including sales tax. If the sales tax rate is 8%, what is the cost of the tennis shoes before the tax is added?

Solution

The total percentage is 100% + 8% = 108%.

$$\begin{aligned} \text{Cost of tennis shoes} &= \frac{\text{Cost with tax included}}{\text{Total percentage}} \\ &= \frac{48.60}{108\%} \\ &= \frac{48.60}{1.08} \\ \text{Use a calculator to divide:} &= 45 \end{aligned}$$

The tennis shoes cost **$45** without tax.

Finding the Percent of Increase or Decrease

When a quantity goes up or down in value, the percent of change can be calculated by comparing the amount of the change to the original amount:

$$\text{Percent of change} = \frac{\text{Amount of change}}{\text{Original amount}} \times 100\%$$

➡ Example

If the price of an item increases from $70 to $84, what is the percent of increase in price?

Solution

The original amount is $70, and the amount of increase is $84 − $70 = $14:

$$\begin{aligned} \text{Percent of change} &= \frac{\text{Amount of increase}}{\text{Original amount}} \times 100\% \\ &= \frac{14}{70} \times 100\% \\ &= \frac{1}{5} \times 100\% \\ &= 20\% \end{aligned}$$

LESSON 4-1 TUNE-UP EXERCISES

Multiple-Choice

1. By the end of the school year, Terry had passed 80% of his science tests. If Terry failed 4 science tests, how many science tests did Terry pass?

   (A) 12
   (B) 15
   (C) 16
   (D) 18

2. A soccer team has played 25 games and has won 60% of the games it has played. What is the minimum number of additional games the team must win in order to finish the season winning 80% of the games it has played?

   (A) 28
   (B) 25
   (C) 21
   (D) 18

3. In a movie theater, 480 of the 500 seats were occupied. What percent of the seats were NOT occupied?

   (A) 0.4%
   (B) 2%
   (C) 4%
   (D) 20%

4. In a certain mathematics class, the part of the class that are members of the math club is 50% of the rest of that class. The total number of math club members in this class is what percent of the entire class?

   (A) 20%
   (B) 25%
   (C) $33\frac{1}{3}\%$
   (D) 50%

5. After 2 months on a diet, John's weight dropped from 168 pounds to 147 pounds. By what percent did John's weight drop?

(A) $12\frac{1}{2}\%$

(B) $14\frac{2}{7}\%$

(C) 21%

(D) 25%

6. If 1 cup of milk is added to a 3-cup mixture that is $\frac{2}{5}$ flour and $\frac{3}{5}$ milk, what percent of the 4-cup mixture is milk?

(A) 80%

(B) 75%

(C) 70%

(D) 65%

7. If the result of increasing a by 300% of a is b, then a is what percent of b?

(A) 20%

(B) 25%

(C) $33\frac{1}{3}\%$

(D) 40%

8. After a 20% increase, the new price of a radio is \$78.00. What was the original price of the radio?

(A) \$15.60

(B) \$60.00

(C) \$62.40

(D) \$65.00

9. After a discount of 15%, the price of a shirt is \$51. What was the original price of the shirt?

(A) \$44.35

(B) \$58.65

(C) \$60.00

(D) \$64.00

10. Three students use a computer for a total of 3 hours. If the first student uses the computer 28% of the total time, and the second student uses the computer 52% of the total time, how many minutes does the third student use the computer?

(A) 24
(B) 30
(C) 36
(D) 42

11. In an opinion poll of 50 men and 40 women, 70% of the men and 25% of the women said that they preferred fiction to nonfiction books. What percent of the number of people polled preferred to read fiction?

(A) 40%
(B) 45%
(C) 50%
(D) 60%

12. In a factory that manufactures light bulbs, 0.04% of all light bulbs manufactured are defective. On the average, there will be three defective light bulbs out of how many manufactured?

(A) 2,500
(B) 5,000
(C) 7,500
(D) 10,000

13. A used-car lot has 4-door sedans, 2-door sedans, sports cars, vans, and jeeps. Of these vehicles, 40% are 4-door sedans, 25% are 2-door sedans, 20% are sports cars, 10% are vans, and 20 of the vehicles are jeeps. If this car lot has no other vehicles, how many vehicles are on the used-car lot?

(A) 300
(B) 400
(C) 480
(D) 600

14. Jack's weight first increased by 20% and then his new weight decreased by 25%. His final weight is what percent of his beginning weight?

(A) 95%
(B) 92.5%
(C) 90%
(D) 88.5%

15. The price of a stock falls 25%. By what percent of the new price must the stock price rise in order to reach its original value?

(A) 25%
(B) 30%
(C) $33\frac{1}{3}\%$
(D) 40%

VOTING POLL	
Candidate A	30%
Candidate B	50%
Undecided	20%

16. The table above summarizes the results of an election poll in which 4,000 voters participated. In the actual election, all 4,000 of these people voted, and those people who chose a candidate in the poll voted for that candidate. People who were undecided voted for candidate A in the same proportion as the people who cast votes for candidates in the poll. Of the people polled, how many voted for candidate A in the actual election?

(A) 1,420
(B) 1,500
(C) 1,640
(D) 1,680

17. A car starts a trip with 20 gallons of gas in its tank. The car traveled at an average speed of 65 miles per hour for 3 hours and consumed gas at a rate of 30 miles per gallon. What percent of the gas in the tank was used for the 3-hour trip?

(A) 32.5
(B) 33.0

(C) 33.5
(D) 34.0

Grid-In

1. A store offers a 4% discount if a consumer pays cash rather than paying by credit card. If the cash price of an item is \$84.00, what is the credit-card purchase price of the same item?

2. During course registration, 28 students enroll in a certain college class. After three boys are dropped from the class, 44% of the class consists of boys. What percent of the original class did girls comprise?

3. A high school tennis team is scheduled to play 28 matches. If the team wins 60% of the first 15 matches, how many additional matches must the team win in order to finish the season winning 75% of its scheduled matches?

4. In a club of 35 boys and 28 girls, 80% of the boys and 25% of the girls have been members for more than 2 years. If n percent of the club have been members for more than 2 years, what is the value of n?

LESSON 4-2 RATIO AND VARIATION

OVERVIEW

A **ratio** is a comparison by division of two quantities that are measured in the same units. For example, if Mary is 16 years old and her brother Gary is 8 years old, then Mary is 2 times as old as Gary. The ratio of Mary's age to Gary's age is 2 : 1 (read as "2 to 1") since

$$\frac{\text{Mary's age}}{\text{Gary's age}} = \frac{16 \text{ years}}{8 \text{ years}} = \frac{2}{1} \text{ or } 2:1$$

One quantity may be related to another quantity so that either the ratio or product of these quantities always remains the same.

RATIO OF *a* TO *b*

The ratio of a to b ($b \neq 0$) is the fraction $\frac{a}{b}$, which can be written as $a : b$ (read as "a to b").

➡ Example

The ratio of the number of girls to the number of boys in a certain class is 3 : 5. If there is a total of 32 students in the class, how many girls are in the class?

Solution

Since the number of girls is a multiple of 3 and the number of boys is the same multiple of 5, let

$$3x = \text{the number of girls in the class}$$

$$\text{and } 5x = \text{the number of boys in the class}$$

Then

$$3x + 5x = 32$$
$$8x = 32$$
$$x = \frac{32}{8} = 4$$

The number of girls = $3x = 3(4) = 12$.

RATIO OF *A* TO *B* TO *C*

If $A : B$ represents the ratio of A to B and $B : C$ represents the ratio of B to C, then the ratio of A to C is $A : C$, provided that B stands for the same number in both ratios. For example, if the ratio of A to B is 3 : 5 and the ratio of B to C is 5 : 7, then the ratio of A to C is 3 : 7. In this case, B represents the number 5 in both ratios.

➡ Example

If the ratio of A to B is 3 : 5 and the ratio of B to C is 2 : 7, what is the ratio of A to C?

Solution

Change each ratio into an equivalent ratio in which the term that corresponds to B is the same number:

- The ratio of A to B is 3 : 5, so the term corresponding to B in this ratio is 5. The ratio of B to C is 2 : 7, so the term corresponding to B in this ratio is 2.
- The least common multiple of 5 and 2 is 10. You need to change each ratio into an equivalent ratio in which the term corresponding to B is 10.
- Multiplying each term of the ratio 3 : 5 by 2 gives the equivalent ratio 6 : 10. Multiplying each term of the ratio 2 : 7 by 5 gives 10 : 35.
- Since the ratio of A to B is equivalent to 6 : 10 and the ratio of B to C is equivalent to 10 : 35, the **ratio of *A* to *C* is 6 : 35**.

Direct Variation

If two quantities change in value so that their ratio always remains the same, then one quantity is said to vary **directly** with the other. When one quantity varies directly with another quantity, a change in one causes a change in the other in the same direction—both increase or both decrease.

> **MATH REFERENCE FACT**
>
> If y varies *directly* as x, then $y = kx$ or, equivalently, $\frac{y}{x} = k$, where k is a nonzero constant.

➡ Example

If $y = kx$, where k is a constant and $y = 27$ when $x = 18$, what is the value of y when $x = 30$?

Solution

The ordered pairs (18, 27) and (30, y) must satisfy the equation $y = kx$ or, equivalently, $\frac{y}{x} = k$. Since $\frac{27}{18} = k$ and $\frac{y}{30} = k$:

$$\frac{27}{18} = \frac{y}{30}$$
$$\frac{3}{2} = \frac{y}{30}$$
$$2y = 90$$
$$y = \frac{90}{2} = 45$$

The value of y is **45**.

➡ Example

If 28 pennies weigh 42 grams, what is the weight in grams of 50 pennies?

Solution

The number of pennies and their weight vary directly since multiplying one of the two quantities of pennies by a constant causes the other to be multiplied by the same constant. If x represents the weight in grams of 50 pennies, then

$$\frac{\text{Pennies}}{\text{Grams}} = \frac{28}{42} = \frac{50}{x}$$

Cross-multiply:

$$28x = 42(50)$$

$$x = \frac{2100}{28} = 75$$

The weight of 50 pennies is **75** grams.

Inverse Variation

If two quantities change in opposite directions, so that their product always remains the same, then one quantity is said to vary **inversely** with the other.

➡ Example

If $xy = k$, where k is a constant and $y = 21$ when $x = 6$, what is the value of y when $x = 9$?

> **MATH REFERENCE FACT**
>
> If y varies *inversely* as x, then $xy = k$ where k is a nonzero constant.

Solution

The ordered pairs (6, 21) and (9, y) must satisfy the equation $xy = k$. Since $6 \times 21 = k$ and $9 \times y = k$,

$$9y = (6)(21)$$

$$\frac{9y}{9} = \frac{126}{9}$$

$$y = 14$$

The value of y is **14**.

➡ Example

Four workers can build a house in 9 days. How many days would it take 3 workers to build the same house?

Solution

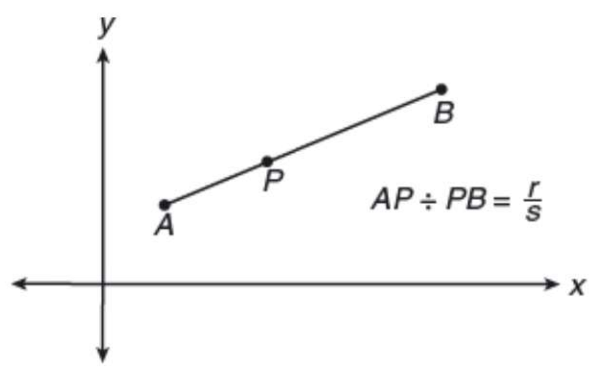
As the number of people working on the house *decreases*, the number of days needed to build the house *increases*. Since this is an inverse variation, the number of workers times the number of days needed to build the house stays constant.

If d represents the number of days that 3 workers take to build the house, then

$$3 \times d = 4 \times 9$$
$$3d = 36$$
$$d = \frac{36}{3} = 12$$

Three people working together would take **12** days to build the house.

DIVIDING A SEGMENT INTO TWO SEGMENTS WITH A GIVEN RATIO



To determine the coordinates of the point P that divides the line segment drawn from point $A(x_A, y_A)$ to point $B(x_B, y_B)$ into two shorter segments such that the ratio of AP to PB is, say r to s:

- Calculate the difference in the x-coordinates ("run") and the difference in the y-coordinates ("rise") from point A to point B.

- Multiply both the run and the rise between the two given points by $\frac{r}{r+s}$ and then add the results to the corresponding coordinates of point A:

$$x\text{-coordinate of point } P:\ x_A + \left(\text{run} \times \frac{r}{r+s}\right)$$

$$y\text{-coordinate of point } P:\ y_A + \left(\text{rise} \times \frac{r}{r+s}\right)$$

➡ Example

What are the coordinates of the point P in the xy-plane that divides the line segment whose endpoints are $A(-1, 4)$ and $B(4, -6)$ into two segments such that the ratio of AP to PB is 2 to 3?

(A) (1, 0)
(B) (0, 2)
(C) (0.5, 1)
(D) (2.5, −3)

Solution

- Between points A and B, the run is $4 - (-1) = 5$ and the rise is $-6 - 4 = -10$.
- Multiply the run and the rise by $\frac{2}{2+3}\left(=\frac{2}{5}\right)$ and then add the products to the corresponding coordinates of point A:

$$P\left(-1+\frac{2}{5}\times 5, 4+\frac{2}{5}\times(-10)\right) = P(-1+2, 4-4) = P(1,0)$$

The correct choice is **(A)**.

MATH REFERENCE FACTS

The probability that an event will happen can be represented by a number from 0 to 1.

- If an event is certain to happen, its probability is 1.
- If an event is an impossibility, its probability is 0.

- $P(not\ E) = 1 - P(E)$. If the probability that it will rain is 20% or 0.2, then the probability it will *not* rain is $1 - 0.2 = 0.8$ or 80%.

SIMPLE PROBABILITY

To find the probability that some event E will happen, write the ratio of the number of favorable outcomes to the total number of different outcomes that are possible:

$$P(E) = \frac{\text{Number of ways } E \text{ can happen}}{\text{Total number of possible outcomes}}$$

The shorthand notation $P(E)$ is used to represent the probability that event E will happen. If 3 yellow marbles and 2 red marbles are placed in a hat and a marble is picked from the hat without looking, then

$$P(\text{yellow}) = \frac{3}{3+2} = \frac{3}{5}$$

LESSON 4-2 TUNE-UP EXERCISES

Multiple-Choice

1. A recipe for 4 servings requires salt and pepper to be added in the ratio of 2 : 3. If the recipe is adjusted from 4 to 8 servings, what is the ratio of the salt and pepper that must now be added?

   (A) 4 : 3
   (B) 2 : 6
   (C) 2 : 3
   (D) 3 : 2

2. On a certain map, $\frac{3}{8}$ of an inch represents 120 miles. How many miles does $1\frac{3}{4}$ inches represent?

   (A) 300
   (B) 360
   (C) 480
   (D) 560

3. The population of a bacteria culture doubles in number every 12 minutes. The ratio of the number of bacteria at the end of 1 hour to the number of bacteria at the beginning of that hour is

   (A) 64 : 1
   (B) 60 : 1
   (C) 32 : 1
   (D) 16 : 1

4. At the end of the season, the ratio of the number of games a team has won to the number of games it lost is 4 : 3. If the team won 12 games and each game played ended in either a win or a loss, how many games did the team play during the season?

   (A) 9
   (B) 15
   (C) 18
   (D) 21

5. If s and t are integers, $8 < t < 40$, and $\frac{s}{t} = \frac{4}{7}$, how many possible values are there for t?

(A) Two
(B) Three
(C) Four
(D) Five

6. A school club includes only sophomores, juniors, and seniors, in the ratio of 1 : 3 : 2. If the club has 42 members, how many seniors are in the club?

(A) 6
(B) 7
(C) 12
(D) 14

7. If $\frac{c - 3d}{4} = \frac{d}{2}$, what is the ratio of c to d?

(A) 5 : 1
(B) 3 : 2
(C) 4 : 3
(D) 3 : 4

8. If 4 pairs of socks costs $10.00, how many pairs of socks can be purchased for $22.50?

(A) 5
(B) 7
(C) 8
(D) 9

9. Two boys can paint a fence in 5 hours. How many hours would it take 3 boys to paint the same fence?

(A) $\frac{3}{2}$
(B) 3
(C) $3\frac{1}{3}$
(D) $4\frac{2}{3}$

10. A car moving at a constant rate travels 96 miles in 2 hours. If the car maintains this rate, how many miles will the car travel in 5 hours?

(A) 480
(B) 240
(C) 210
(D) 192

11. The number of kilograms of corn needed to feed 5,000 chickens is 30 less than twice the number of kilograms needed to feed 2,800 chickens. How many kilograms of corn are needed to feed 2,800 chickens?

(A) 70
(B) 110
(C) 140
(D) 190

12. The number of calories burned while jogging varies directly with the number of minutes spent jogging. If George burns 180 calories by jogging for 25 minutes, how many calories does he burn by jogging for 40 minutes?

(A) 200
(B) 276
(C) 288
(D) 300

13. If y varies directly as x and $y = 12$ when $x = c$, what is y in terms of c when $x = 8$?

(A) $\frac{2c}{3}$

(B) $\frac{3}{2c}$

(C) $20c$

(D) $\frac{96}{c}$

$$\frac{x}{z} = \frac{1}{3}$$

14. If in the equation above x and z are integers, which are possible values of $\frac{x^2}{z}$?

I. $\frac{1}{9}$

II. $\frac{1}{3}$

III. 3

(A) II only
(B) III only
(C) I and III only
(D) II and III only

15. If $a - 3b = 9b - 7a$, then the ratio of a to b is

(A) 3 : 2
(B) 2 : 3
(C) 3 : 4
(D) 4 : 3

16. The ratio of A to B is $a : 8$, and the ratio of B to C is $12 : c$. If the ratio of A to C is 2 : 1, what is the ratio of a to c?

(A) 2 : 3
(B) 3 : 2
(C) 4 : 3
(D) 3 : 4

17. If $8^r = 4^t$, what is the ratio of r to t?

(A) 2 : 3
(B) 3 : 2
(C) 4 : 3
(D) 3 : 4

18. If $\frac{a+b}{b} = 4$ and $\frac{a+c}{c} = 3$, what is the ratio of c to b?

(A) 2 : 3
(B) 3 : 2
(C) 2 : 1
(D) 3 : 1

19. In a certain college, the ratio of mathematics majors to English majors is 3 : 8. If in the following school year the number of mathematics majors increases 20% and the number of English majors decreases 15%, what is the new ratio of mathematics majors to English majors?

(A) 4 : 9
(B) 1 : 2
(C) 9 : 17
(D) 17 : 32

20. At a college basketball game, the ratio of the number of freshmen who attended to the number of juniors who attended is 3 : 4. The ratio of the number of juniors who attended to the number of seniors who attended is 7 : 6. What is the ratio of the number of freshmen to the number of seniors who attended the basketball game?

(A) 7 : 8
(B) 3 : 4
(C) 2 : 3
(D) 1 : 2

21. It took 12 men 5 hours to build an airstrip. Working at the same rate, how many additional men could have been hired in order for the job to have taken 1 hour less?

(A) Two
(B) Three
(C) Four
(D) Six

22. If x represents a number picked at random from the set $\{-3, -2, -1, 0, 1, 2\}$, what is the probability that x will satisfy the inequality $4 - 3x < 6$?

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

23. What are the coordinates of the point P in the xy-plane that divides the segment whose endpoints are $A(-2, 9)$ and $B(7, 3)$ into two segments such that the ratio of AP to PB is 1 to 2?

(A) $P(1, 5)$
(B) $P(4, 1)$
(C) $P(1, 7)$
(D) $P(2, 6)$

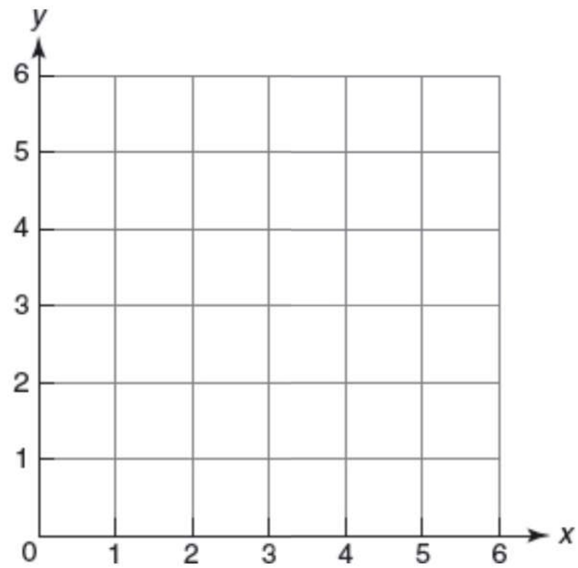
Grid-In

1. A string is cut into 2 pieces that have lengths in the ratio of 2 : 9. If the difference between the lengths of the 2 pieces of string is 42 inches, what is the length in inches of the shorter piece?

2. For integer values of a and b, $b^a = 8$. The ratio of a to b is equivalent to the ratio of c to d, where c and d are integers. What is the value of c when $d = 10$?

3. Jars A, B, and C each contain 8 marbles. What is the minimum number of marbles that must be transferred among the jars so that the ratio of the number of marbles in jar A to the number in jar B to the number in jar C is 1 : 2 : 3?

4. A political campaign organizer has determined that the number of hours needed to get out a mailing for her candidate is inversely related to the number of campaign workers she has. If she uses the information in the accompanying graph, how many hours would it take to do the mailing if 125 workers are used?



5. A square dartboard is placed in the first quadrant from $x = 0$ to 6 and $y = 0$ to 6, as shown in the accompanying figure. A triangular region on the dartboard is enclosed by the graphs of the equations $y = 2$, $x = 6$, and $y = x$ (not shown). Find the probability that a dart that randomly hits the dartboard will land in the triangular region formed by the three lines.

LESSON 4-3 RATE PROBLEMS

OVERVIEW

A ratio of two quantities that have different units of measurement is called a **rate**. For example, if a car travels a total distance of 150 miles in 3 hours, the average rate of speed is the distance traveled divided by the amount of time required to travel that distance:

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}} = \frac{150 \text{ miles}}{3 \text{ hours}} = 50 \text{ miles per hour}$$

Rate problems are usually solved using the general relationship

$$\text{Rate (of } A \text{ per unit } B) \times B = A$$

You are using the correct rate relationship if the units check, as in

$$\underbrace{\text{Rate}}_{\frac{\text{Miles}}{\cancel{\text{Hour}}}} \times \underbrace{\text{Time}}_{\cancel{\text{Hours}}} = \underbrace{\text{Distance}}_{\text{Miles}}$$

UNIT COST PROBLEMS

Unit cost problems require you to figure out a rate by calculating the cost per item. Multiplying this rate by a given number of items gives the total cost of those items.

➡ Example

If 5 cans of soup cost \$1.95, how much do 3 cans of soup cost?

Solution 1

Since 5 cans of soup cost \$1.95, the cost of 1 can is

$$\frac{\$1.95}{5\text{ cans}} = 0.39\text{ dollar per can}$$

To find the cost of 3 cans, multiply the rate of 0.39 dollar per can by 3:

$$\text{Cost of 3 cans} = 3\text{ cans} \times 0.39\frac{\$}{\text{can}} = \$1.17$$

Solution 2

The number of cans of soup varies directly with the cost of the cans. Form a proportion in which *x* represents the cost of 3 cans of soup:

$$\begin{aligned} \frac{\text{Cost}}{\text{Number of cans}} = \frac{x}{3} &= \frac{1.95}{5} \\ 5x &= 3(1.95) \\ x &= 5.85 \\ x &= \frac{5.85}{5} = 1.17 \end{aligned}$$

The cost of 3 cans of soup is **$1.17**.

MOTION PROBLEMS

The solution of motion problems depends on the relationship

$$\text{Rate} \times \text{Time} = \text{Distance}$$

➥ Example

John rode his bicycle to town at the rate of 15 miles per hour. He left the bicycle in town for minor repairs and walked home along the same route at the rate of 3 miles per hour. Excluding the time John spent in taking the bike into the repair shop, the trip took 3 hours. How many hours did John take to walk back?

Solution

Since the trip took a total of 3 hours,

let x = the number of hours John took to ride his bicycle to town,

and $3 - x$ = the number of hours John took to walk back from town.

	Rate ×	Time =	Distance
To town	15 mph	x hours	$15x$
Return trip	3 mph	$3 - x$ hours	$3(3 - x)$

Since John traveled over the same route, the two distances must be equal. Hence,

$$\begin{aligned} 15x &= 3(3 - x) \\ 15x &= 9 - 3x \\ 18x &= 9 \\ x &= \frac{9}{18} = \frac{1}{2} \text{ hour} \end{aligned}$$

John took $3 - \frac{1}{2} = 2\frac{1}{2}$ hours to walk back.

WORK PROBLEMS

When solving a problem involving two or more people or pieces of machinery working together to complete a job, use these relationships:

- The rate of work is the reciprocal of the time it takes to complete the entire job.
- Rate of work × Actual time worked = Fractional part of job completed

Suppose John can mow an entire lawn in 3 hours and Maria can mow the same lawn in 4 hours. The number of hours, x, it would take them to mow the lawn working together is given by the equation:

$$\overbrace{\left(\frac{1}{3}\right)x}^{\text{Part of job completed by John}} + \overbrace{\left(\frac{1}{4}\right)x}^{\text{Part of job completed by Maria}} = 1 \text{ whole job}$$

➥ Example

A new printing press can print 5,000 flyers in half the amount of time it takes for an older printing press to print the same 5,000 flyers. Working together, the two printing presses can complete the entire job in 3 hours. How long would it take the faster printing press working alone to complete the job?

Solution

If x represents the amount of time it takes the slower printing press to complete the job working alone, then the faster printing press can complete the job in $\frac{x}{2}$ hours. Hence,

$$\left(\frac{1}{x}\right)3+\left(\frac{1}{x/2}\right)3=1$$
$$\left(\frac{3}{x}\right)+\left(\frac{6}{x}\right)=1$$
$$3+6=x$$

Since the slower printing press would take 9 hours working alone to complete the job, the faster press would take $\frac{9}{2}$ = 4.5 hours working alone to complete the job.

If the original equation in the previous example was written in the equivalent form $\frac{1}{x}+\frac{2}{x}=\frac{1}{3}$, then

- $\frac{1}{x}$ represents the part of the whole job completed by the slower press in 1 hour.
- $\frac{2}{x}$ represents the part of the whole job completed by the faster press in 1 hour.
- $\frac{1}{3}$ represents the part of the whole job that is completed in 1 hour with both presses working together.

LESSON 4-3 TUNE-UP EXERCISES

Multiple-Choice

1. If four pens cost $1.96, what is the greatest number of pens that can be purchased for $7.68?

   (A) 12
   (B) 14
   (C) 15
   (D) 16

2. Two pipes of different diameters may be used to fill a swimming pool. The pipe with the larger diameter working alone can fill the swimming pool 1.25 times faster than the other pipe when it works alone. One hour after the larger pipe is opened, the smaller pipe is opened, and the swimming pool is filled 5 hours later. Which equation could be used to find the number of hours, x, it would take for the larger pipe to fill the pool working alone?

   (A) $\left(\frac{1}{1.25x}\right)5+\left(\frac{1}{x}\right)6=1$
   (B) $\left(\frac{1}{x}\right)5+\left(\frac{1}{1.25x}\right)6=1$
   (C) $\left(\frac{x}{5}\right)1.25+\left(\frac{x}{6}\right)=1$
   (D) $\left(\frac{x}{5}\right)+\left(\frac{x}{6}\right)1.25=1$

3. On a certain map, 1.5 inches represent a distance of 120 miles. If two cities on this map are 1 foot apart, what is the distance, in miles, between the cities?

   (A) 180
   (B) 480
   (C) 960
   (D) 1,080

4. A freight train left a station at 12 noon, going north at a rate of 50 miles per hour. At 1:00 P.M. a passenger train left the same station, going south

at a rate of 60 miles per hour. At what time were the trains 380 miles apart?

(A) 3:00 P.M.
(B) 4:00 P.M.
(C) 4:30 P.M.
(D) 5:00 P.M.

5. A man drove to work at an average rate of speed of 60 miles per hour and returned over the same route driving at an average rate of speed of 40 miles per hour. If his total driving time was 1 hour, what was the total number of miles in the round trip?

(A) 12
(B) 24
(C) 30
(D) 48

6. If x people working together at the same rate can complete a job in h hours, what part of the same job can one person working alone complete in k hours?

(A) $\frac{k}{xh}$

(B) $\frac{h}{xk}$

(C) $\frac{k}{x+h}$

(D) $\frac{kh}{x}$

7. An electrician can install 5 light fixtures in 3 hours. Working at that rate, how long will it take the electrician to install 8 light fixtures?

(A) $3\frac{4}{5}$ hours

(B) $4\frac{1}{5}$ hours

(C) $4\frac{1}{2}$ hours

(D) $4\frac{4}{5}$ hours

8. A freight train and a passenger train start toward each other at the same time from two towns that are 500 miles apart. After 3 hours, the trains are still 80 miles apart. If the average rate of speed of the passenger train is 20 miles per hour faster than the average rate of speed of the freight train, what is the average rate of speed, in miles per hour, of the freight train?

(A) 40
(B) 45
(C) 50
(D) 60

9. One machine can seal 360 packages per hour, and an older machine can seal 140 packages per hour. How many MINUTES will the two machines working together take to seal a total of 700 packages?

(A) 48
(B) 72
(C) 84
(D) 90

10. A motor boat traveling at 18 miles per hour traveled the length of a lake in one-quarter of an hour less time than it took when traveling at 12 miles per hour. What was the length in miles of the lake?

(A) 6
(B) 9
(C) 12
(D) 15

11. Carmen went on a trip of 120 miles, traveling at an average of x miles per hour. Several days later she returned over the same route at a rate that was 5 miles per hour faster than her previous rate. If the time for the return trip was one-third of an hour less than the time for the outgoing trip, which equation can be used to find the value of x?

(A) $\frac{120}{x+5} = \frac{1}{3}$

(B) $\frac{x}{120} = \frac{x+5}{120} - \frac{1}{3}$

(C) $\frac{120}{x+(x+5)} = \frac{1}{3}$

(D) $\frac{120}{x} = \frac{120}{x+5} + \frac{1}{3}$

12. Jonathan drove to the airport to pick up his friend. A rainstorm forced him to drive at an average speed of 45 miles per hour, reaching the airport in 3 hours. He drove back home at an average speed of 55 miles per hour. How long, to the *nearest tenth of an hour*, did the trip home take him?

    (A) 2.0 hours
    (B) 2.5 hours
    (C) 2.8 hours
    (D) 3.7 hours

13. A plumber works twice as fast as his apprentice. After the plumber has worked alone for 3 hours, his apprentice joins him and working together they complete the job 4 hours later. How many hours would it have taken the plumber to do the entire job by himself?

    (A) 9
    (B) 12
    (C) 14
    (D) 18

Grid-In

1. Fruit for a dessert costs $1.20 a pound. If 5 pounds of fruit are needed to make a dessert that serves 18 people, what is the cost of the fruit needed to make enough of the same dessert to serve 24 people?

2. A printing press produces 4,600 flyers per hour. At this rate, in how many *minutes* can the same printing press produce 920 flyers?

FOREIGN CURRENCY CONVERSIONS

U.S. Dollar to British Pound = 1.56 to 1
British Pound to Euro = 1 to 1.38

3. Foreign currency conversion rates for the British pound, U.S. dollar, and Euro are listed above. What would be the cost in U.S. dollars for a shirt that has a purchase price of 46 Euros, correct to the *nearest dollar*?

4. Joseph typed a 1,200-word essay in 25 minutes with an average of 240 words on a page. At this rate, how many 240-word pages can he type in 1 hour?

LESSON 4-4 CONVERTING UNITS OF MEASUREMENT

OVERVIEW

Changing from one unit of measurement to another requires using a fractional conversion factor that tells the relationship between the two units so that like units cancel as in

$$2 \text{ miles} = 2 \cancel{\text{miles}} \times \underbrace{5{,}280 \frac{\text{feet}}{\cancel{\text{mile}}}}_{\substack{\text{conversion factor} \\ \text{miles to feet}}} = 10{,}560 \text{ feet}$$

TIP

A conversion factor expresses the relationship between the two units as a fraction and is always numerically equal to 1. For example, because the conversion factor of $\frac{5{,}280 \text{ feet}}{1 \text{ mile}}$ has the same distance in both its numerator and its denominator, it has a value of 1.

HOW TO CONVERT UNITS

To convert from one unit of measurement to another:

- Write the conversion factor as a fraction placing the old units in the denominator and the new units in the numerator. For example, to change from feet (old units) into an equivalent number of miles (new units), use $\frac{1 \text{ mile}}{5{,}280 \text{ feet}}$ as the conversion factor.
- Multiply the number of units you want to convert by the conversion factor.
- Cancel any units that are in both a numerator and a denominator. Then check that the answer is in the correct units.

➡ Example

If an object is moving at an average rate of speed of $18\frac{\text{km}}{\text{min}}$, how many meters does it travel in 5 seconds?

Solution

Convert $18\frac{\text{km}}{\text{min}}$ to an equivalent number of meters per second. Then multiply the result by 5 seconds:

- Since 1 kilometer = 1,000 meters, to convert from kilometers to meters use the conversion factor $\frac{1{,}000\text{ m}}{1\text{ km}}$. Thus,

$$18\frac{\text{km}}{\text{min}} = 18\frac{\cancel{\text{km}}}{\text{min}} \times \frac{1{,}000\text{ meters}}{1\ \cancel{\text{km}}} = 18{,}000\frac{\text{meters}}{\text{min}}$$

- The numerator for the conversion factor for time must contain minutes so that it cancels the minutes in the denominator of $18{,}000\frac{\text{m}}{\text{min}}$. Hence, use the conversion factor of $\frac{1\text{ min}}{60\text{ sec}}$:

$$\begin{aligned} 18{,}000\frac{\text{m}}{\text{min}} &= \frac{18{,}000\text{ m}}{1\ \cancel{\text{min}}} \times \frac{1\ \cancel{\text{min}}}{60\text{ sec}} \\ &= \frac{18{,}000\text{ m}}{60\text{ sec}} \\ &= 300\frac{\text{m}}{\text{sec}} \end{aligned}$$

- Find the distance:

$$300\frac{\text{m}}{\text{sec}} \times 5\text{ sec} = 1{,}500\text{ m}$$

➡ Example

The average download speed for Max's computer's Internet connection is 30 megabits per second. Assuming no interruptions in Internet service, what is the best estimate for the maximum number of complete video files that Max can download to his computer in a six-hour period if each video file is 4.2

gigabytes in size? [1 megabyte = 8 megabits and 1 gigabyte = 1,024 megabytes]

(A) 15
(B) 18
(C) 19
(D) 21

TIP

Write the conversion factors with their units in fractional form as in the preceding examples. When converting between units, like units that appear in a numerator and in a denominator are canceled so that the final answer has the correct unit of measurement. If units do *not* cancel as expected, you have used a wrong conversion factor and may need to invert it so that the corresponding units cancel.

Solution

- Convert from $\frac{\text{megabits}}{\text{sec}}$ to $\frac{\text{megabits}}{\text{hour}}$:

$$30\,\frac{\text{megabits}}{\text{sec}} = 30\,\frac{\text{megabits}}{\cancel{\text{sec}}} \times \frac{60\ \cancel{\text{sec}}}{1\ \cancel{\text{min}}} \times \frac{60\ \cancel{\text{min}}}{1\ \text{hour}} = 108{,}000\,\frac{\text{megabits}}{\text{hour}}$$

- Convert from $\frac{\text{megabits}}{\text{hour}}$ to $\frac{\text{gigabytes}}{\text{hour}}$:

$$108{,}000\,\frac{\text{megabits}}{\text{hour}} = 108{,}000\,\frac{\cancel{\text{megabits}}}{\text{hour}} \times \frac{1\ \cancel{\text{megabyte}}}{8\ \cancel{\text{megabits}}} \times \frac{1\ \text{gigabyte}}{1024\ \cancel{\text{megabytes}}} \approx 13.184\,\frac{\text{gigabytes}}{\text{hour}}$$

- In a 6-hour period, the total number of gigabytes that can be downloaded is

$$13.184\,\frac{\text{gigabytes}}{\cancel{\text{hour}}} \times 6\ \cancel{\text{hours}} = 79.104\ \text{gigabytes}$$

Since $79.104 \div 4.2 \approx 18.8$, a maximum of 18 video files can be download. The correct choice is **(B)**.

SCIENTIFIC NOTATION

When a very large or very small number includes a sequence of trailing or leading zeros, particularly in science measurements, it is sometimes convenient to write that number as a decimal number between 1 and 10 times a power of 10.

- If the original number is greater than 10, the number of places the decimal point needs to be moved to the *left* becomes the power of 10. If necessary, insert the decimal point in the original number, as in

$$32{,}000{,}000. = 3.2 \times 10^{7}$$

7 places

- If the original number is between 0 and 1, make the power of 10 negative followed by the number of places the decimal point needs to be moved to the *right*, as in

$$0.0000608 = 6.08 \times 10^{-5}$$

5 places

LESSON 4-4 TUNE-UP EXERCISES

Multiple-Choice

1. Which expression could be used to change 8 kilometers per hour to meters per minute?

   (A) $\frac{8\text{ km}}{\text{hr}} \times \frac{1\text{ km}}{1{,}000\text{ m}} \times \frac{1\text{ hr}}{60\text{ min}}$

   (B) $\frac{8\text{ km}}{\text{hr}} \times \frac{1{,}000\text{ m}}{1\text{ km}} \times \frac{60\text{ min}}{1\text{ hr}}$

   (C) $\frac{8\text{ km}}{\text{hr}} \times \frac{1{,}000\text{ m}}{1\text{ km}} \times \frac{1\text{ hr}}{60\text{ min}}$

   (D) $\frac{8\text{ km}}{\text{hr}} \times \frac{1\text{ km}}{1{,}000\text{ m}} \times \frac{60\text{ min}}{1\text{ hr}}$

2. Which expression represents 72 kilometers per hour expressed as meters per hour?

   (A) 7.2×10^{-2}
   (B) 7.2×10^{2}
   (C) 7.2×10^{-3}
   (D) 7.2×10^{4}

3. If the mass of a proton is 1.67×10^{-24} gram, what is the number of grams in the mass of 1,000 protons?

   (A) 1.67×10^{-27}
   (B) 1.67×10^{-23}
   (C) 1.67×10^{-22}
   (D) 1.67×10^{-21}

4. There are 12 players on a basketball team. Before a game, both ankles of each player are taped. Each roll of tape will tape three ankles. Which product can be used to determine the number of rolls of tape needed to tape the players' ankles?

(A) $12\text{ players}\cdot\frac{1\text{ player}}{2\text{ ankles}}\cdot\frac{3\text{ ankles}}{1\text{ roll}}$

(B) $12\text{ players}\cdot 2\text{ ankles}\cdot\frac{3\text{ rolls}}{1\text{ ankle}}$

(C) $12\text{ players}\cdot\frac{2\text{ ankles}}{1\text{ player}}\cdot\frac{1\text{ roll}}{3\text{ ankles}}$

(D) $12\text{ players}\cdot\frac{1\text{ roll}}{3\text{ players}}\cdot\frac{3\text{ ankles}}{\text{roll}}$

$$\frac{40\text{ yd}}{4.5\text{ sec}}\cdot\frac{3\text{ ft}}{1\text{ yd}}\cdot\frac{5{,}280\text{ ft}}{1\text{ mi}}\cdot\frac{60\text{ sec}}{1\text{ min}}\cdot\frac{60\text{ min}}{1\text{ hr}}$$

5. A sprinter who can run the 40-yard dash in 4.5 seconds converts his speed into miles per hour, using the product above. Which fraction in the product is *incorrectly* written to convert his speed?

(A) $\frac{3\text{ ft}}{1\text{ yd}}$

(B) $\frac{5{,}280\text{ ft}}{1\text{ mi}}$

(C) $\frac{60\text{ sec}}{1\text{ min}}$

(D) $\frac{60\text{ min}}{1\text{ hr}}$

6. A star constellation is approximately 3.1×10^4 light years from Earth. One light year is about 5.9×10^{12} miles. What is the approximate distance, in miles, between Earth and the constellation?

(A) 1.83×10^{17}
(B) 9.0×10^{49}
(C) 1.9×10^8
(D) 9.0×10^{16}

7. An eye medication that is used to treat increased pressure inside the eye is packaged in 2.5 milliliter bottles. During the manufacturing process, a 10 decaliter capacity bin is used to fill the bottles. If 1 decaliter is equivalent to 10 liters and 1 liter is equivalent to 1,000 milliliters, what is the maximum number of bottles that can be filled?

(A) 4×10^5
(B) 4×10^4

(C) 2.5×10^3
(D) 2.5×10^2

Grid-In

1. At a party, six 1-liter bottles of soda are completely emptied into 8-ounce cups. What is the *least* number of cups that are needed? [There are approximately 1.1 quarts in 1 liter.]

2. On a certain map, 1 inch represents 2 kilometers. A region is located on the map that is 1.5 inches by 4.0 inches. What is the actual area of the region in square *miles* if 1 kilometer is equal to 0.6 mile?

3. The distance from Earth to Mars is 136,000,000 miles. A spacecraft travels at an average speed of 28,500 kilometers per hour. Determine, to the *nearest day*, how long it will take the spacecraft to reach Mars. [1 kilometer = 0.6 miles]

4. A certain generator will run for 1.5 hours on one liter of gas. If the gas tank has the shape of a rectangular box that is 25 cm by 20 cm by 16 cm, how long will the generator run on a full tank of gas? [1 liter = 1,000 cubic centimeters]

5. One knot is one nautical mile per hour, and one nautical mile is 6,080 feet. If a cruiser ship has an average speed of 3.5 knots, how many feet does the ship travel in 24 minutes?

LESSON 4-5 LINEAR AND EXPONENTIAL FUNCTIONS

OVERVIEW

The SAT assumes that you are familiar with the characteristics of two important classes of functions: the linear function and the exponential function.

- In a linear function, the greatest power of the variable is 1, as in $y = 3x + 2$. A key distinguishing feature of a linear function is that the rate of change between the variables is *constant*.
- In an exponential function, the variable is in an exponent, as in $y = 2^x$. The rate of change between the variables is *not* constant but changes by a fixed percent of its previous value as when interest on a savings account is compounded year after year.

LINEAR FUNCTIONS

The graph of a linear function is a straight line. When a linear function is written in the form $f(x) = mx + b$, m is the slope of the line and b is its y-intercept. If $y = f(x) = 3x + 2$, then each time x increases by 1 unit, y increases by 3 units, so the rate of change is fixed at $\frac{3}{1}$ or 3. Thus, the slope of a line represents the rate at which a linear function changes in value with respect to x. Linear functions can be used to represent situations that involve *constant* rates of change.

➥ Example

A certain 4 inch spring stretches 1.5 inches for each ounce of weight attached to it.

Solution

The length of the spring, L, when an x-ounce weight is attached to it, can be represented by the function $L(x) = 4 + 1.5x$.

➡ Example

A car starts a trip with 20 gallons of gasoline in its tank and consumes gas at a rate of 1 gallon for each 25 miles traveled.

Solution

After traveling x miles, the amount of gas, g, remaining in the tank is $g(x) = 20 - \frac{1}{25}x$.

INTERPRETING THE PARTS OF A LINEAR MODEL

When a linear equation of the form $y = a + bx$ is used to represent or model a real-world situation, the values of the constants a and b have specific meanings within the particular setting described. Typically,

- The constant a represents some starting value for y or some initial condition.
- The constant b tells for each unit change in x how much y increases or decreases.

If the dollar value, L, of a certain laptop computer decreases m months after its purchase according to the equation $L = 2{,}100 - 28m$, then -28 represents the number of dollars the value of the laptop *decreases* each month after its purchase, whereas 2,100 represents the starting or purchase price of the laptop.

➡ Example

When there are x milligrams of a certain drug in a patient's bloodstream, a patient's heart rate, h, in beats per minute, can be modeled by the equation $h = 70 + 0.5x$. Which statements are true?

I. 10 minutes after taking the drug, the patient's heart rate increases to 75 beats per minute.

II. When the drug is not in the bloodstream, the patient's heart rate is 70 beats per minute.

III. For each 10 milligram increase of the drug in the patient's bloodstream, the patient's heartbeat increases 5 beats per minute.

(A) I and II
(B) I and III
(C) II and III
(D) I, II, and III

Solution

- I. The model does not explicitly consider time so Statement I is *not* correct.
- II. When $x = 0$, $h = 70 + 0.5(0) = 70$ so Statement II is true.
- III. Statement III is true since each time x increases by 10, h increases by $0.5 \times 10 = 5$ beats per minute.

The correct choice is **(C)**.

GRAPHS AS MODELS

A graph of a function can model a situation by giving a visual picture of how one real-world quantity measured along the vertical y-axis depends on another quantity measured along the horizontal x-axis. For example, the graph in Figure 4.1 shows how the cost C, in dollars, of manufacturing a certain product depends on the number of units manufactured.

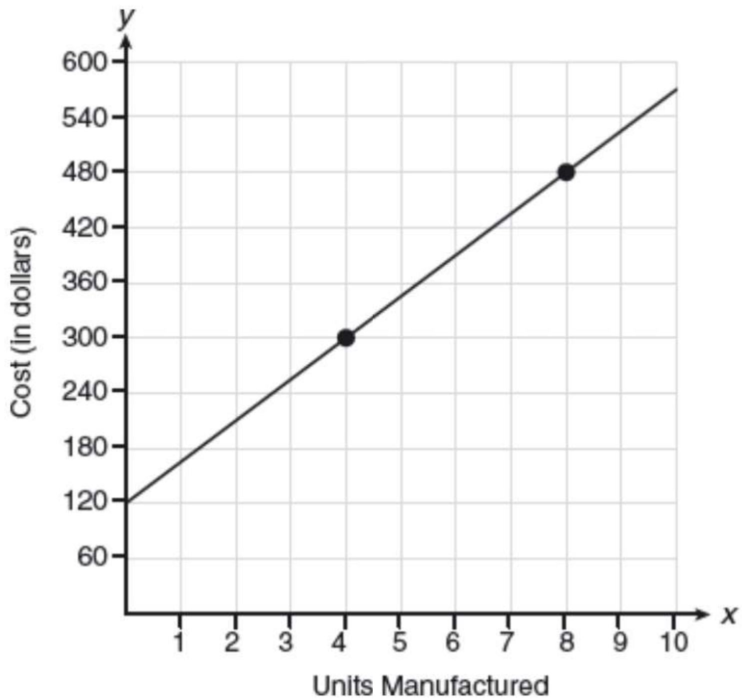


Figure 4.1 Cost as a function of the number of units manufactured

Since the graph is a straight line, the rate at which the cost changes for each additional unit manufactured is constant and equal to the slope of the line.

> **TIP**
>
> **If the relationship between two quantities is represented by a line, then you can determine the rate of change between these quantities by finding the slope of the line.**

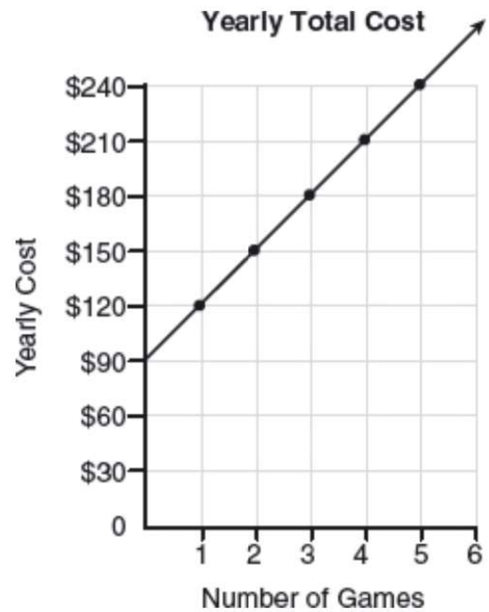
- To find the slope of the line, choose any two convenient points with integer coordinates such as (4, 300) and (8, 480):

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{480-300}{8-4} = \frac{180}{4} = 45\frac{\text{dollars}}{\text{unit}}$$

Thus, the cost of manufacturing each additional unit is **$45**.

- To find an equation of the line from the graph, write the general form of the equation as $C = ax + b$ where a is the slope of the line and b is the y-intercept. Because $a = 45$ and $b = 120$, the equation of the line is $\boldsymbol{C = 45x + 120}$.

➡ Example



The graph represents the yearly cost of playing 0 to 5 games of golf at the Sunnybrook Golf Course, which includes the yearly membership fee.

a. What is the cost of playing one game of golf?

b. Write a linear function that expresses the total cost in dollars, C, of joining the club and playing n games during the year. What is the cost of playing 10 games of golf?

Solution

a. The slope of the line represents the cost per game of golf. Pick any two points on the line with integer coordinates such as (1, 120) and (2, 150) and find the slope of the line:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{150-120}{2-1} = 30\,\frac{\text{dollars}}{\text{game}}$$

The cost of one game of golf is **$30**.

b. When the number of games is 0, the cost is $90. Hence, the y-intercept of the line represents the yearly cost of membership. Because the slope of the line is 30 and its y-intercept is 90, an equation of the line is $C =$ $\mathbf{30}n + \mathbf{90}$. To find the cost of playing 10 games, substitute 10 for n in C

$= 30n + 90$, which gives $C = 30 \cdot 10 + 90 = 390$. Hence, the total cost of joining the club and playing 10 games during the year is **$390**.

MATH REFERENCE FACT

In a linear function, y changes at a constant rate—each time x changes by 1 unit, the amount y changes stays the same. In an exponential function of the form $y = Ab^x$, the rate at which y changes is *not* constant—each time x increases by 1 unit, y increases to b times its previous value.

THE EXPONENTIAL FUNCTION

An **exponential function** is an equation of the form $y = Ab^x$, where $A \neq 0$ and b is a positive constant other than 1. The constant A represents the initial amount of y which for successive integer values of x (= 1, 2, 3, 4, ...) is repeatedly multiplied by the base b. Figure 4.2 shows the effect of b on the graph of $y = b^x$.

- If $b > 1$, then as x increases, y increases and the graph rises at an increasingly faster rate.
- If $0 < b < 1$, then as x increases, y decreases and the graph falls but at a slower and slower rate.

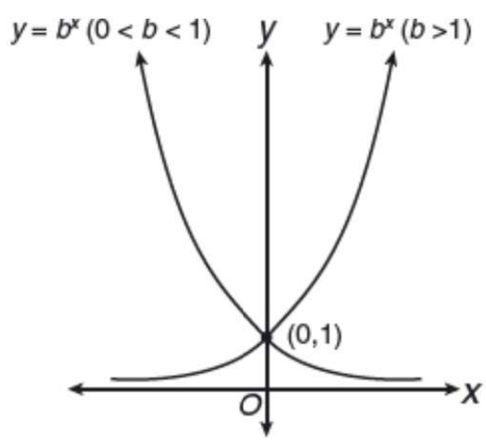
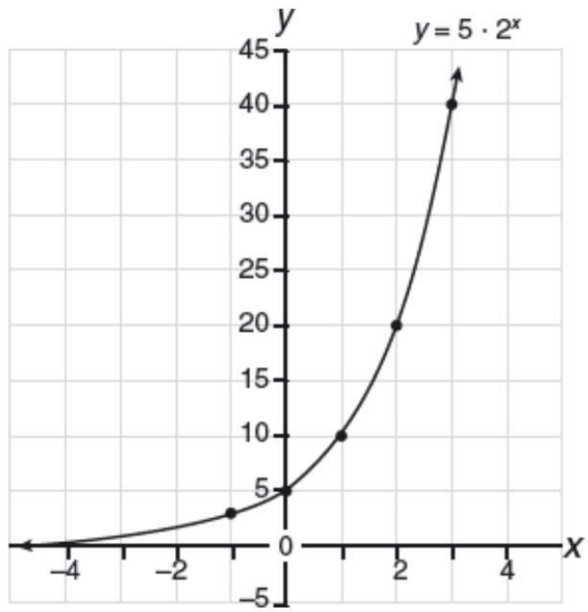


Figure 4.2 The exponential function $y = b^x$

Illustrating Exponential Change Graphically

The slope of a line is constant across the line. The slope or the *rate* at which y changes along an exponential curve. The accompanying table of values represent points from the graph of $y = 5 \cdot 2^x$ shown in Figure 4.3.

x	y
–1	2.5
0	5
1	10
2	20
3	40



You should verify that each time x increases by 1 in Figure 4.3, y increases to twice its previous value. In terms of percent, y increases by 100% of its previous value each time x increases by 1. This type of behavior represents exponential *growth*.

Illustrating Exponential Change Algebraically

Suppose \$320 is invested in an account that earns 7% interest compounded annually.

- After one year, the balance in the account is

$$\underbrace{320}_{\text{initial amount}} + \underbrace{320 \times 0.07}_{\text{interest}} = 320(1+0.07) = 320\underbrace{(1.07)}_{\substack{\uparrow \\ \text{multiplying factor}}}$$

- After the second year, the balance in the account is

$$\underbrace{320(1.07)}_{\text{old amount}}+\underbrace{[320(1.07)]\times(0.07)}_{\text{interest}}=320(1.07)(1+0.07)=320\underbrace{(1.07)}_{x}^{2}$$

- After n years, the balance, y, has increased by a factor of 1.07 for a total of n times. Thus,

$$y=\underbrace{320(1.07)(1.07)\cdots(1.07)}_{n\text{ factors}}=320\underbrace{(1.07)}_{x}^{n}$$

If function f represents the balance after n years, then $f(n) = 320(1.07)^n$. This exponential function has a starting amount of 320 and a growth, or multiplying, factor of 1.07. Many real-life processes involve either exponential growth or exponential decay. Exponential *growth* occurs when the multiplying factor is greater than 1. Exponential *decay* happens if the multiplying factor is a positive number less than 1. Here are more examples of exponential change:

- Assume a certain population of 1,000 insects triples every 6 days. To represent this process as an exponential function, make a table with the first few terms and then generalize:

Number of Days	Population
6	1000×3
12	$(1000 \times 3) \times 3 = 1000 \times 3^2$
18	$(1000 \times 3 \times 3) \times 3 = 1000 \times 3^3$
...	...
n	$1000 \times 3^{\frac{n}{6}}$

Thus, the exponential function $f(n) = 1000 \times 3^{\frac{n}{6}}$ describes this growth process, where n is the number of days that have elapsed.

- In the geometric sequence

400, 200, 100, 50, 25, ...

each term after the first is obtained by multiplying the term that comes before it by $\frac{1}{2}$.

To represent this process as an exponential function, make a table with the first few terms and then generalize:

Term Number	Term
1	400
2	$400 \times \frac{1}{2}$
3	$\left(400 \times \frac{1}{2}\right) \times \frac{1}{2} = 400 \times \left(\frac{1}{2}\right)^2$
4	$\left(400 \times \frac{1}{2} \times \frac{1}{2}\right) \times \frac{1}{2} = 400 \times \left(\frac{1}{2}\right)^3$
...	...
n	$400 \times \left(\frac{1}{2}\right)^{n-1}$

Thus, the exponential function $a(n) = 400 \times \left(\frac{1}{2}\right)^{n-1}$ describes this decay process, where $a(n)$ represents the nth term of this number sequence.

GENERAL FORMULAS FOR EXPONENTIAL GROWTH AND DECAY

If an initial quantity A changes exponentially at a rate of r% per time period, then after x successive time periods, the amount present, y, is given by the formulas in the accompanying table.

Process	Exponential Function
Growth	$y = A(1 + r)^x$
Decay	$y = A(1 - r)^x$

In an exponential function of the form $y = Ab^x$ where $b = 1 \pm r$:

- when $b > 1$, the initial amount A is *increasing* exponentially at a rate of r%. If a colony of insects begins with 20 insects and increases exponentially in number at a rate of 35% every 5 days, then after 30 days the number of insects, y, is

$$y = 20(1 + 0.35)^{\frac{30}{5}} = 20(1.35)^6$$

- when $0 < b < 1$, the initial amount A is *decreasing* exponentially at a rate of r%. If the population of a town with a current population of 5,400 is decreasing exponentially at a rate of 13% per year, then after 8 years the population, y, is

$$y = 5{,}400(1 - 0.13)^8 = 5{,}400(0.87)^8$$

- The quantity b that is being raised to a power is sometimes referred to as the **growth factor**.

➡ Example

The current population of a town is 10,000. If the population, P, increases by 3.5% every six months, which equation could be used to find the population after t years?

(A) $P = 10{,}000(1.035)^{\frac{t}{2}}$
(B) $P = 10{,}000\,(0.965)^{2t}$
(C) $P = 10{,}000\,(1.035)^{2t}$
(D) $P = 10{,}000(0.965)^{\frac{t}{2}}$

Solution

Since the growth factor is $1 + 3.5\% = 1 + 0.035 = 1.035$, eliminate choices (B) and (D). Because the growth cycle is every 6 months, there are 2 growth cycles or compounding periods each year in which the population increases by 3.5% over its current amount, $P = 10{,}000\,(1.035)^{2t}$.

The correct choice is **(C)**.

➡ Example

A car loses its value at a rate of 4.5% annually. If a car is purchased for \$24,500, which equation can be used to determine the value of the car, V, after 5 years?

(A) $V = 24{,}500\ (0.045)^5$
(B) $V = 24{,}500\ (0.55)^5$
(C) $V = 24{,}500\ (1.045)^5$
(D) $V = 24{,}500\ (0.955)^5$

Solution

Since the growth factor is $1 - 4.5\% = 1 - 0.045 = 0.955$, $V = 24{,}500(0.955)^5$.

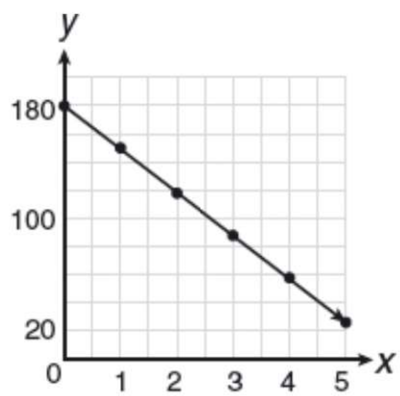
The correct choice is **(D)**.

LESSON 4-5 TUNE-UP EXERCISES

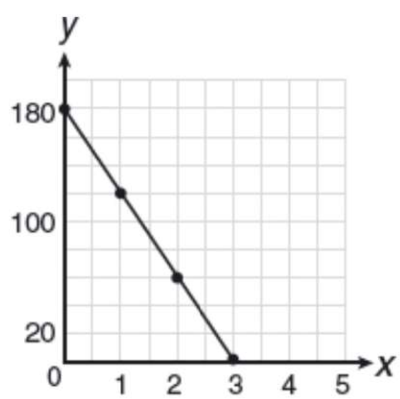
Multiple-Choice

1. On January 1, a share of a certain stock costs $180. Each month thereafter, the cost of a share of this stock decreased by one-third. If x represents the time, in months, and y represents the cost of the stock, in dollars, which graph best represents the cost of a share over the following 5 months?

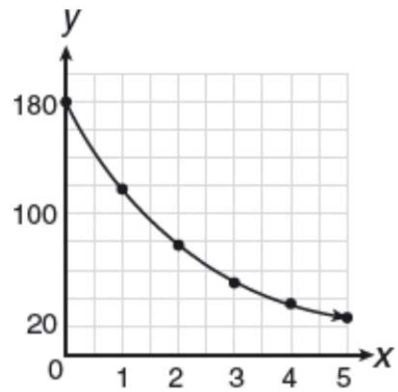
   (A) Graph A
   (B) Graph B
   (C) Graph C
   (D) Graph D



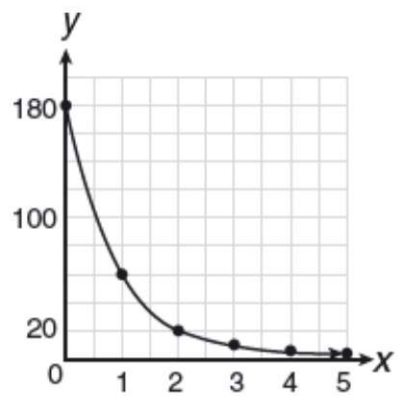
Graph A



Graph B



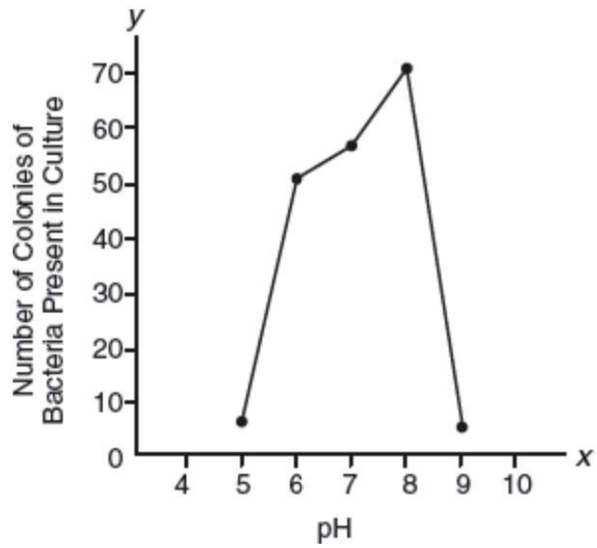
Graph C



Graph D

2. A certain population of insects starts at 16 and doubles every 6 days. What is the population after 60 days?

(A) 2^6
(B) 2^{10}
(C) 2^{14}
(D) 2^{32}



3. The accompanying graph illustrates the presence of a certain strain of bacteria at various pH levels. Between which two pH values is the rate at which the number of colonies of bacteria increasing at the lowest rate?

(A) 5 to 6
(B) 6 to 7
(C) 7 to 8
(D) 8 to 9

4. After a single sheet of paper is folded in half, there are two layers of paper. The same sheet of paper is repeatedly folded in half. If function f represents the number of layers of paper that results when the original sheet of paper is folded a total of x times, then which equation could represent this function?

(A) $f(x) = 2x$
(B) $f(x) = x^2$
(C) $f(x) = 2^x$
(D) $f(x) = 4^{\frac{x}{2}}$

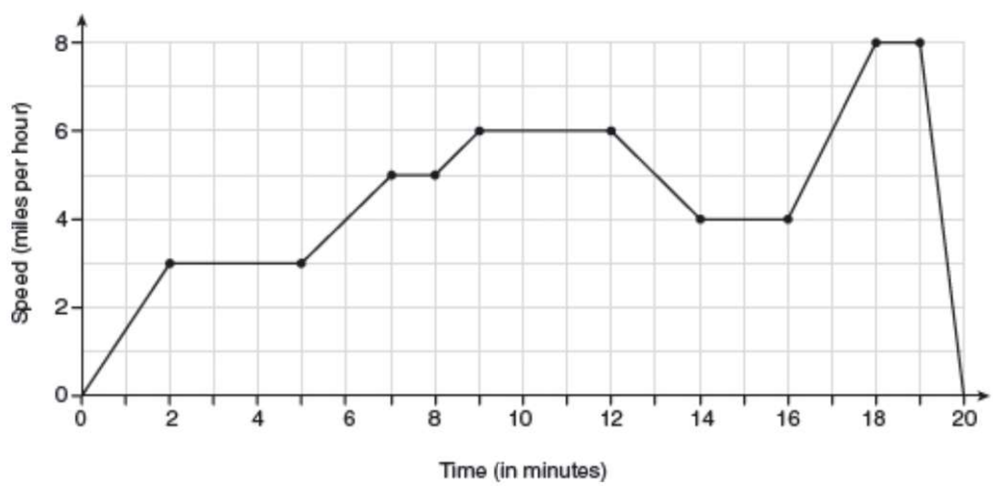
5. The yearly growth in the number of fast food restaurants in a certain retail chain can be modeled by the function $f(n) = 5 + 8n$. According to this model which statement is true?

(A) 8 is the initial number of restaurants; 5 is the number of restaurants added each year after the first year.
(B) 5 is the starting number of restaurants; 8 is the number of restaurants added each year after the first year.
(C) The retail chain opened with 13 restaurants.
(D) The y-intercept of the graph of function f shows the year in which the retail chain made a zero profit.

$$C = 60 + 0.05d$$

6. The equation above represents the total monthly cost, C, in dollars of a data plan offered by a cell phone company when the data usage in a month exceeds a 1 gigabyte limit by d megabytes. According to the model, what is the meaning of 0.05?

(A) The cost per megabyte of data used
(B) The cost per gigabyte of data used
(C) The cost per megabyte of data after one gigabyte of data is used
(D) The cost of each additional gigabyte of data after the first megabyte of data is used



7. The graph above represents a jogger's speed during her 20-minute jog around her neighborhood. Which statement best describes what the jogger was doing during the 9 to 12 minute interval of her jog?

(A) She was standing still.
(B) She was increasing her speed.
(C) She was decreasing her speed.

(D) She was jogging at a constant rate.

8. If a_n represents the nth term of the sequence 54,18,6, ..., and a_1 represents the first term, then $a_n =$

(A) $6\left(\frac{1}{3}\right)^n$

(B) $6\left(\frac{1}{3}\right)^{n-1}$

(C) $54\left(\frac{1}{3}\right)^n$

(D) $54\left(\frac{1}{3}\right)^{n-1}$

9. The owner of a small computer repair business has one employee, who is paid an hourly rate of $22. The owner estimates his weekly profit using the function $P(x) = 8{,}600 - 22x$. In this function, x represents the number of

(A) computers repaired per week.
(B) employee's hours worked per week.
(C) customers served per week.
(D) days worked per week.

10. The breakdown of a sample of a chemical compound is represented by the function $p(n) = 300(0.5)^n$, where $p(n)$ represents the number of milligrams of the substance that remains at the end of n years. Which of the following is true?

I. 300 represents the number of milligrams of the substance that remains after the first year.
II. 0.5 represents the fraction of the starting amount by which the substance gets reduced by the end of each year.
III. Each year the substance gets reduced by one-half of 300.

(A) I only
(B) II only
(C) I and III only
(D) II and III only

11. Some banks charge a fee on savings accounts that are left inactive for an extended period of time. The equation $y = 5{,}000(0.98)^x$ represents the value, y, of one account that was left inactive for a period of x years. What is the y-intercept of this equation and what does it represent?

(A) 0.98, the percent of money in the account initially
(B) 0.98, the percent of money in the account after x years
(C) 5,000, the amount of money in the account initially
(D) 5,000, the amount of money in the account after x years

12. Chris plans to purchase a car that loses its value at a rate of 14% per year. If the intial cost of the car is $27,000, which of the following equations should Chris use to determine the value, v, of the car after 4 years?

(A) $v = 27{,}000\ (1.14)^4$
(B) $v = 27{,}000\ (0.14)^4$
(C) $v = 27{,}000\ (0.86)^4$
(D) $v = 27{,}000\ (0.86 \times 4)$

13. Vanessa plans to invest $10,000 for 5 years at an annual interest rate of 6% compounded annually. Which equation could be used to determine the profit, P, Vanessa earns from her initial investment?

(A) $P = 10{,}000 \times (1.06)^5$
(B) $P = 10{,}000 \times [(1.06)^5 - 1]$
(C) $P = 10{,}000 \times [(1.06)^5 + 1]$
(D) $P = 10{,}000 \times [5(1.06) - 1]$

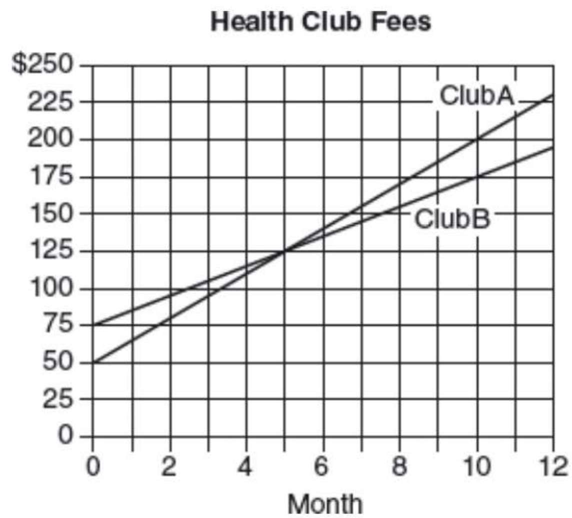
14. Miriam and Jessica are growing bacteria in a laboratory. Miriam uses the growth function $f(t) = n^{2t}$ to model her experiment while Jessica uses the function $g(t) = \left(\frac{n}{2}\right)^{4t}$ to model her experiment. In each case, n represents the initial number of bacteria, and t is the time, in hours. If Miriam starts with 16 bacteria, how many bacteria should Jessica start with to achieve the same growth over time?

(A) 32
(B) 16
(C) 8

(D) 4

15. The number of square units, A, in the area covered by a bacteria culture is increasing at a rate of 20% every 7 days. If the bacteria culture initially covers an area of 10 square centimeters, which equation can be used to find the number of square units in the area covered by the bacteria culture after d days?

(A) $A = 10(1.20)^{\frac{d}{7}}$
(B) $A = 10(1.20)^{7d}$
(C) $A = 10(0.80)^{\frac{d}{7}}$
(D) $A = (12)^{7d}$



16. Two health clubs offer different membership plans. The accompanying graph represents the total yearly cost of belonging to Club A and Club B for one year. The yearly cost includes a membership fee plus a fixed monthly charge. By what amount does the monthly charge for Club A exceed the monthly charge for Club B?

(A) $5.00
(B) $7.50
(C) $10.00
(D) $12.50

x	y
0.5	9.0

1	8.75
1.5	8.5
2	8.25
2.5	8.0

17. Based on the data in the table above, which statement is true about the rate of change of y with respect to x?

    (A) It is constant and equal to $-\frac{1}{8}$.
    (B) It is constant and equal to 2.
    (C) It is constant and equal to $-\frac{1}{2}$.
    (D) It is not constant.

18. The City Tunnel and Bridge Authority in a certain city estimates that 40,000 vehicles currently travel over a certain bridge per year but, due to highway construction and changing traffic patterns, vehicle traffic over the bridge will begin to decline by 12% every 5 years. Which of the following expressions best represents the vehicle traffic projections for this bridge n years from now?

    (A) $40{,}000(0.12)^{\frac{n}{5}}$
    (B) $40{,}000(0.88)^{5n}$
    (C) $40{,}000(0.12)^{5n}$
    (D) $40{,}000(0.88)^{\frac{n}{5}}$

19. Which of the accompanying tables that show how population is changing over time illustrate exponential decay?

Time (months)	Population
0	10,000
6	7,000
12	4,000
18	1,000

Table I

Time (months)	Population
0	10,000
6	5,000
12	2,500
18	1,250

Table II

Time (months)	Population
0	10,000
6	20,000
12	40,000
18	80,000

Table III

Time (months)	Population
0	10,000
6	15,000
12	20,000
18	25,000

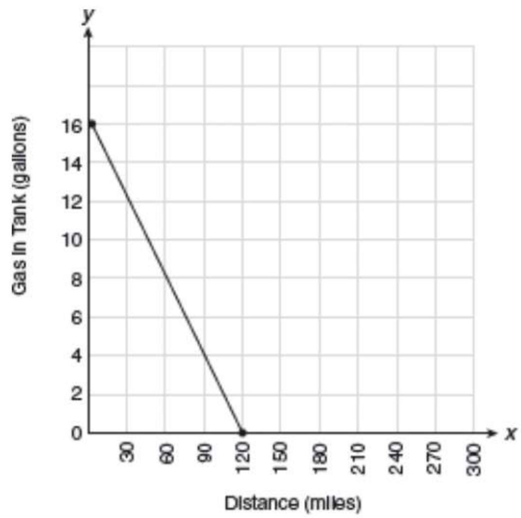
Table IV

(A) Table I
(B) Table II
(C) Table III
(D) Table IV

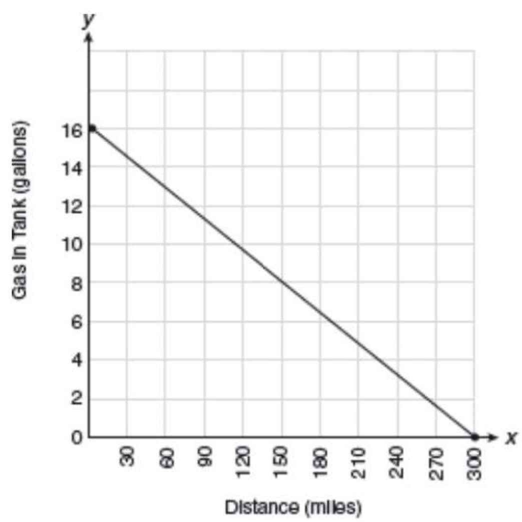
20. A radioactive substance has an initial mass of 100 grams, and its mass is reduced by 40% every 5 years. Which equation could be used to find the number of grams in the mass, y, that remains after x years?

(A) $y = 100(0.4)^{5x}$
(B) $y = 100(0.6)^{5x}$
(C) $y = 100(0.4)^{\frac{x}{5}}$
(D) $y = 100(0.6)^{\frac{x}{5}}$

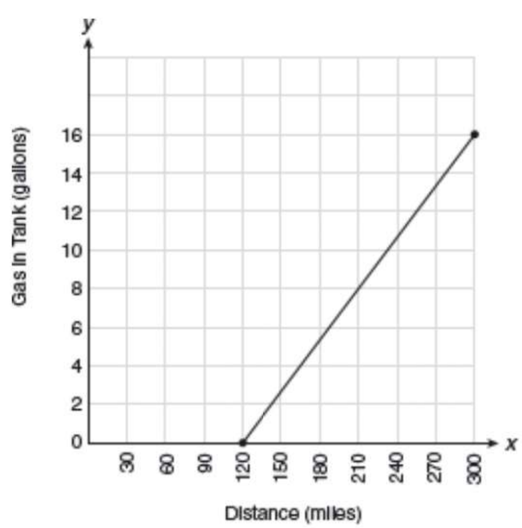
21. The gas tank in a car holds a total of 16 gallons of gas. The car travels 75 miles on 4 gallons of gas. If the gas tank is full at the beginning of a trip, which graph represents the rate of change in the amount of gas in the tank?



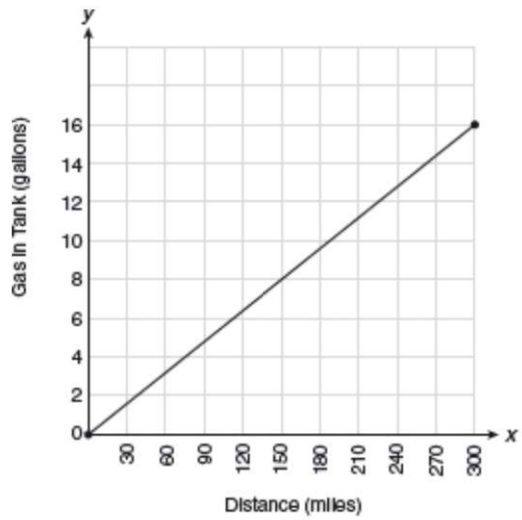
Graph A



Graph B



Graph C



Graph D

(A) Graph A
(B) Graph B
(C) Graph C
(D) Graph D

Grid-In

1. Jacob begins painting at 12:00 noon. At 12:30 P.M. he estimates that 13 gallons of paint are left, and at 3:30 he estimates that 4 gallons of paint remain. If the paint is being used at a constant rate, how many gallons of paint did Jacob have when he started the job?

2. The number of hours, H, needed to manufacture X computer monitors is given by the function $H = kX + q$, where k and q are constants. If it takes 270 hours to manufacture 100 computer monitors and 410 hours to manufacture 160 computer monitors, how many *minutes* are required to manufacture each additonal computer monitor?

3. Given a starting population of 100 bacteria, the formula $b(t) = 100(2^t)$ can be used to determine the number of bacteria, b, after t periods of time. If each time period is 15 minutes long, how many minutes will it take for the population of bacteria to reach 51,200?

4. A certain drug raises a patient's heart rate, h, in beats per minute, according to the equation $h(x) = 65 + 0.2x$, where x is the number of

milligrams of the drug in the patient's bloodstream. After t hours, the level of the drug in the patient's bloodstream decreases exponentially according to the equation $x(t) = 512(0.7)^t$. After 5 hours, what is the number of beats per minute in the patient's heart rate, correct to the *nearest whole number*?

5. The breakdown of a sample of a chemical compound is represented by the function $p(t) = 300\left(\frac{1}{2}\right)^t$, where $p(t)$ represents the number of milligrams of the substance, and t represents the time, in years. If $t = 0$ represents the year 2015, what will be the first year in which the amount of the substance remaining falls to less than 5 milligrams?

6. Sasha invested $1,200 in a savings account at an annual interest rate of 1.6% compounded annually. She made no further deposits or withdrawals. To the *nearest dollar*, how much more money did she have in the account after 3 years than after 2 years?

LESSON 4-6 GRAPHS AND TABLES

OVERVIEW

Graphs and tables are used to visually summarize data in a way that is concise and easy to read. Before attempting to answer questions based on a graph, you should quickly scan the graph to get an overview of what data and information are being presented. Pay close attention to the type of graph and its title. Take note of any descriptive labels and units of measurement along horizontal and vertical sides of a bar or line graph. If a circle graph is given, look for the whole amount that the circle represents.

COMMON TYPES OF GRAPHS

Recognizing the type of graph may help you answer questions about the graph. In general,

- **A circle graph** or pie chart shows how the parts that comprise a whole compare with the whole and to each other. Each "slice" of a pie chart is typically labeled with the percent that part is of the whole. The sum of the "slices" of a pie chart add up to 100% as shown in Figure 4.4.

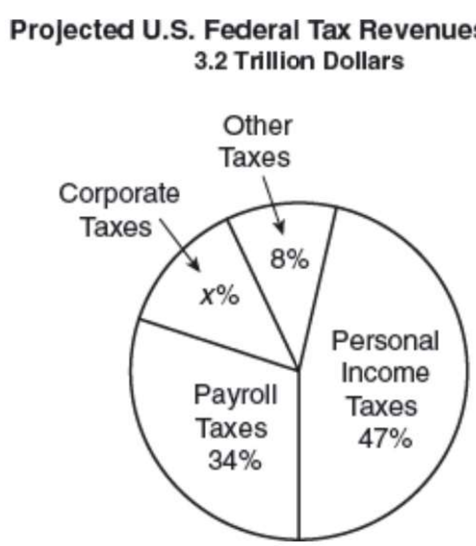


Figure 4.4 Circle graph (pie chart)

Since $x + 47 + 34 + 8 = 100$, you can conclude that the amount of tax revenue from corporate tax sources is 11% of 3.2 trillion dollars, which is 0.352 trillion dollars or, equivalently, 352 billion dollars.

- **A bar graph** compares similar categories of items using rectangular bars. The base of each bar has a nonnumerical label that describes a category. The height or length of each bar represents a numerical amount associated with that category. Figure 4.5 summarizes the number of different colleges to which five high school seniors applied for admission.

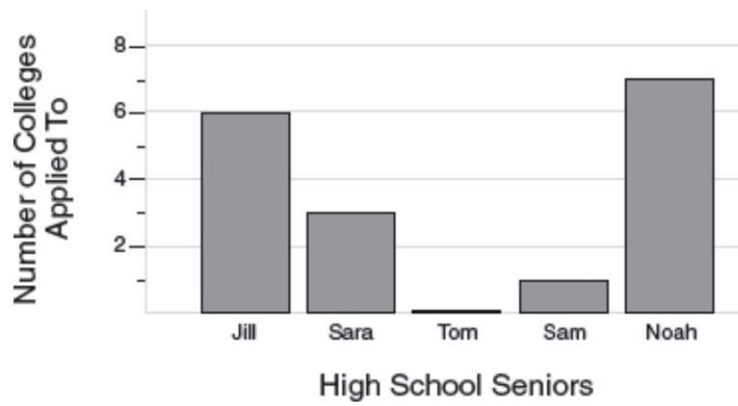


Figure 4.5 High School Seniors

The average number of colleges applied to per student is

$$\frac{6+3+0+1+5}{5} = \frac{15}{5} = 3\text{ colleges per student}$$

- A **histogram** is similar to a bar graph except that the base of each bar of a histogram is a single *numerical* value or an interval of values. The length of the bar is the corresponding amount of data items that have that numerical value or that fall into that interval of values. The descriptive labels along the axes, as well as the title of the graph, explain what the numerical quantities represent. Figure 4.6 shows a histogram that summarizes the number of colleges applied to for 18 high school seniors selected at random.

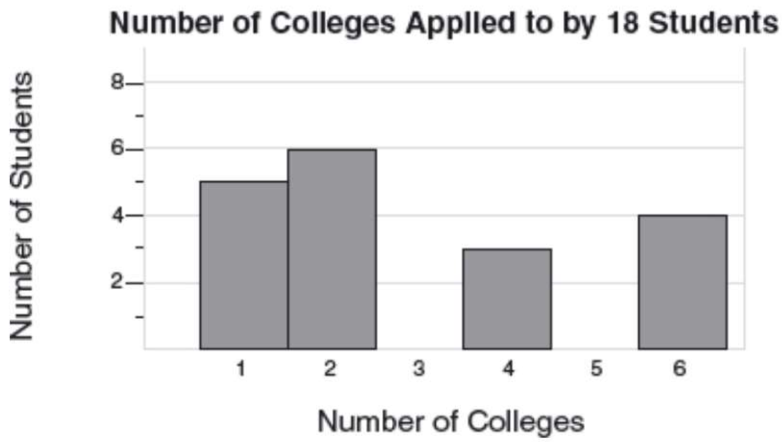


Figure 4.6 Number of Colleges

Notice that there are 5 students who applied to one college, 6 students who applied to two colleges, and so forth. The average number of colleges applied to per student is

$$\frac{(5\times1)+(6\times2)+(0\times3)+(3\times4)+(0\times5)+(4\times6)}{14}=\frac{53}{18}\approx 3\text{colleges per student.}$$

- **A line graph** is used to represent how one data variable changes with another particularly when one of the variables is time. For example, a line graph could be used to show how the price of a stock rises and falls over time, as shown in Figure 4.7.

Figure 4.7 Line graph of the price of a stock from January through May

If a line segment slants up in a time interval, as in January in Figure 4.7, the amount is increasing for that time interval. If a line segment falls in a time interval, as in April, the amount is decreasing in that interval. A horizontal line segment indicates no change, as in February. Since over the 5-month period the price of the stock increases from $15 to $75, the average increase per month of the stock is $\frac{\$75-\$15}{5} = \$12$ per month.

TWO-WAY TABLES AND CONDITIONAL PROBABILITY

A **two-way table** displays the set of all possible pairings of the values of *two* different categories of data. The values for one of these variables are listed in a vertical column and the values of the other are placed along a horizontal row as illustrated in the accompanying table, which shows the results of a survey of the types of books downloaded by 140 individuals according to age group. Since the horizontal row for the 22–50 age group intersects the vertical column for action novels in the cell that contains the number 10, you know that the age group 22–50 downloaded 10 action novels to their tablets. The totals for each vertical column and each horizontal row are given in the last cells for each column and row. The sum of the row totals **equals** the sum of the column totals.

	Type of Book Downloaded to a Computer Tablet				
Age Group	Non-Fiction	Action Novel	Romance Novel	Science Fiction Novel	Total
Under 21	2	14	15	7	38
22–50	12	10	19	18	59
Over 50	14	13	11	5	43
Total	28	37	45	30	140

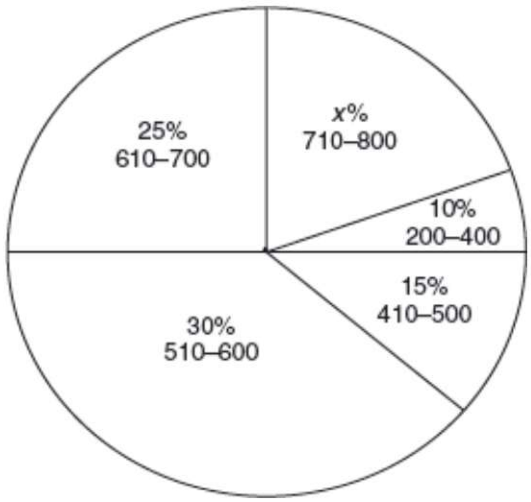
Based on the table, you should be able to answer questions such as these:

- What percent of the individuals surveyed downloaded a novel? Since 28 people downloaded a non-fiction book, 140 – 28 = 112 downloaded some type of novel. Thus, $\frac{112}{140} = 0.8 = 80\%$ of those surveyed, downloaded a novel.
- If a person over 21 is selected at random, what is the probability the person downloaded either a non-fiction book or a romance novel? Of the 140 persons surveyed, 12 + 14 = 26 individuals over 21 years of age downloaded a non-fiction book and 19 + 11 = 30 individuals over 21 years of age downloaded a romance novel. Hence, the probability asked for is $\frac{26 + 30}{140} = \frac{56}{140} = \frac{2}{5}$.

LESSON 4-6 TUNE-UP EXERCISES

Multiple-Choice

SAT Math Scores at Cedar Lane High School



Questions 1 and 2 refer to the graph above.

1. If there are 72 SAT Math scores between 510 and 600, how many SAT Math scores are above 700?

   (A) 40
   (B) 48
   (C) 56
   (D) 64

2. If 20% of the students with SAT Math scores from 610 to 700 received college scholarships, how many students with SAT Math scores from 610 to 700 received college scholarships?

   (A) 12
   (B) 18
   (C) 30
   (D) 48

Minimum Age Requirement (years)	Number of States

14	7
15	12
16	27
17	2
18	2

Questions 3 and 4 refer to the table above.

3. The table above shows the minimum age requirement for obtaining a driver's license. In what percent of the states can a person obtain a driver's license before the age of 16?

(A) 94%
(B) 47%
(C) 38%
(D) 19%

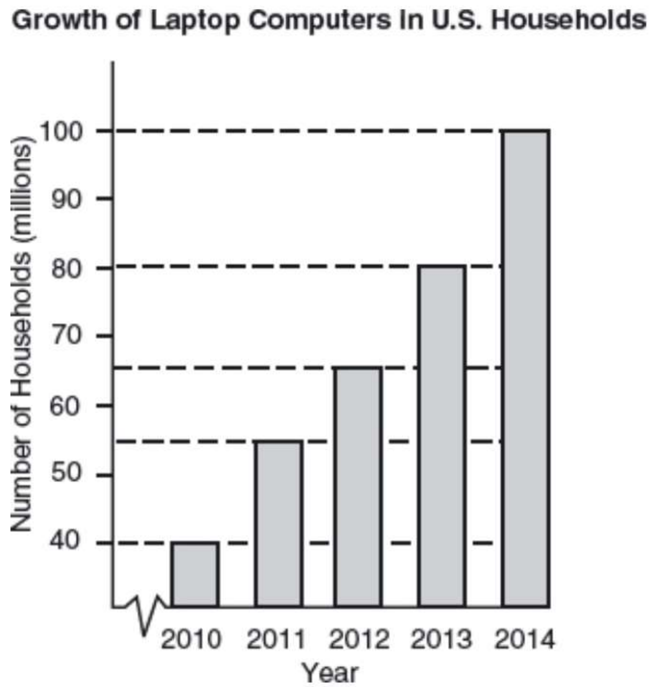
4. If a state is chosen at random, what is the probability that the minimum age for obtaining a driver's license in that state will be at least 16?

(A) $\frac{1}{25}$

(B) $\frac{2}{25}$

(C) $\frac{19}{50}$

(D) $\frac{31}{50}$



Questions 5 and 6 refer to the graph above.

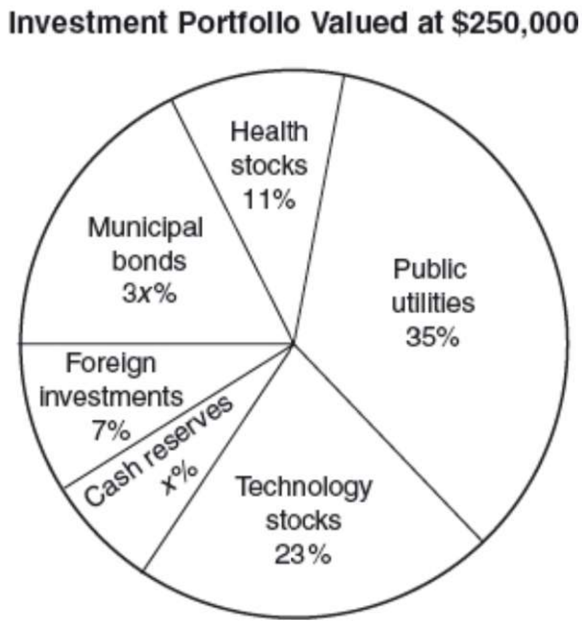
The graph above shows the number of U.S. households with laptop computers for the years 2010 to 2014.

5. What was the percent of increase in the number of households with laptops from 2010 to 2014?

   (A) 40%
   (B) 60%
   (C) 120%
   (D) 150%

6. The greatest percent of increase in the number of households with laptops occurred in which two consecutive years?

   (A) 2010 to 2011
   (B) 2011 to 2012
   (C) 2012 to 2013
   (D) 2013 to 2014

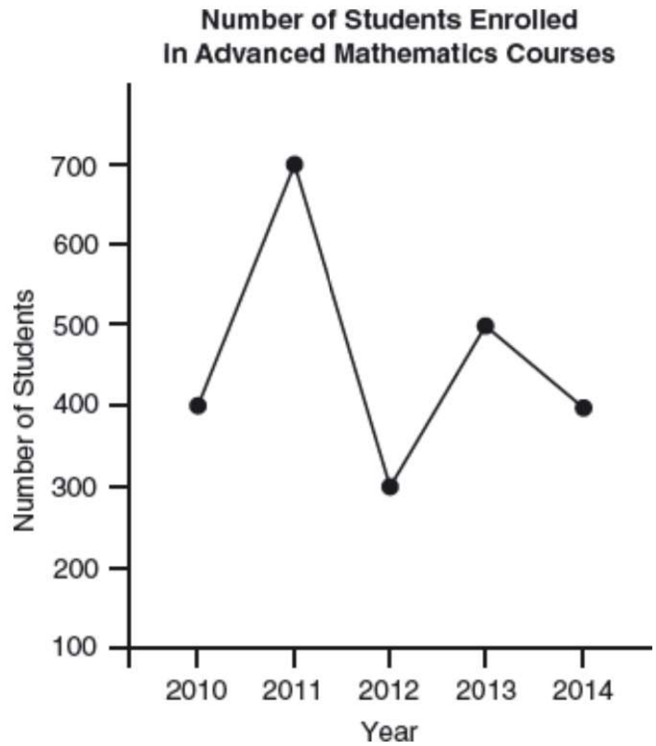


Questions 7 and 8 refer to the graph above.

The graph above shows how $250,000 is invested.

7. How much money is invested in municipal bonds?
   (A) $45,000
   (B) $37,500
   (C) $35,000
   (D) $30,000

8. After 20% of the amount that is invested in technology stocks is reinvested in health stocks, how much money is invested in health stocks?
   (A) $77,500
   (B) $65,000
   (C) $45,000
   (D) $39,000



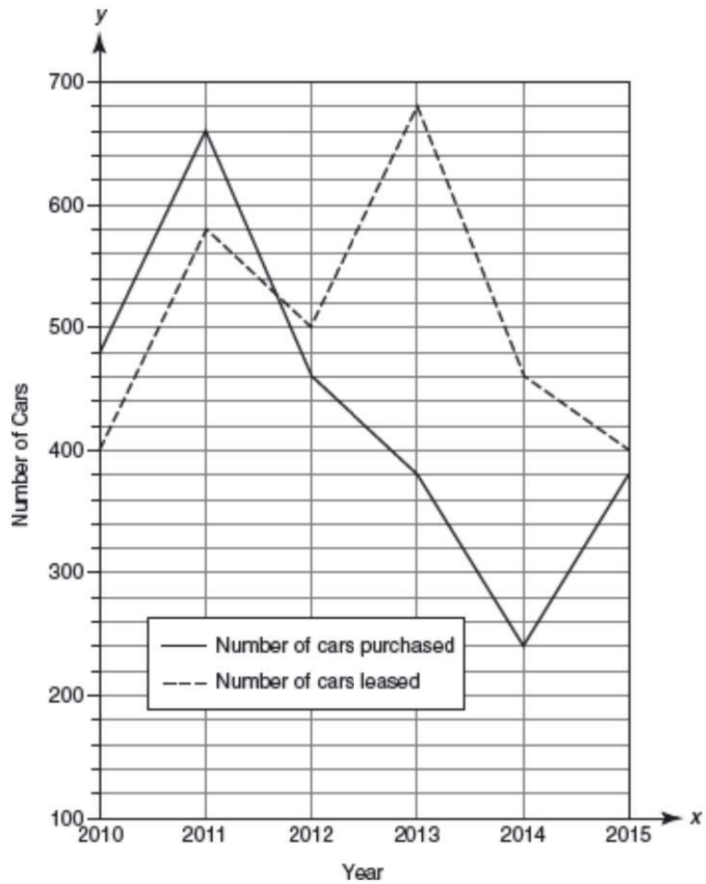
Questions 9 and 10 refer to the graph above.

9. The percent increase in the number of students enrolled in advanced mathematics courses from 2010 to 2011 exceeded the percent increase from 2012 to 2013 by approximately what percent?

(A) 133
(B) 75
(C) 67
(D) 8

10. From 2014 to 2015 the number of students enrolled in advanced mathematics courses increased by the same percent that student enrollment in advanced mathematics courses dropped from 2013 to 2014. What was the approximate number of students enrolled in advanced mathematics courses in 2015?

(A) 420
(B) 440
(C) 450
(D) 480



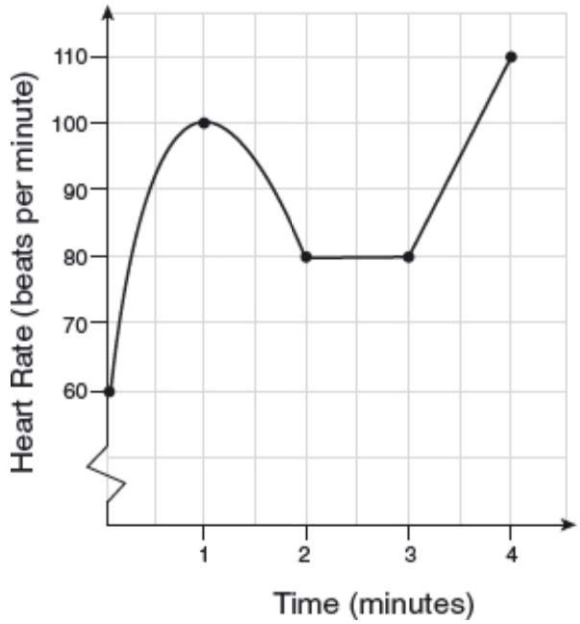
Questions 11 and 12 refer to the graph above.

11. In 2012, the number of cars purchased was *x* percent of the number of cars leased. What is the best approximation for *x*?

(A) 75
(B) 80
(C) 85
(D) 90

12. Which of the following is the best approximation for the decrease in the number of cars purchased per year between 2011 and 2014?

(A) 105
(B) 140
(C) 300
(D) 420



Questions 13 and 14 refer to the graph above which shows the heart rate, in beats per minute, of a jogger during a 4-minute interval.

13. Over what interval of time, in minutes, was the jogger's heart rate changing at a constant rate?

(A) 0 to 1
(B) 1 to 2
(C) 2 to 3
(D) 3 to 4

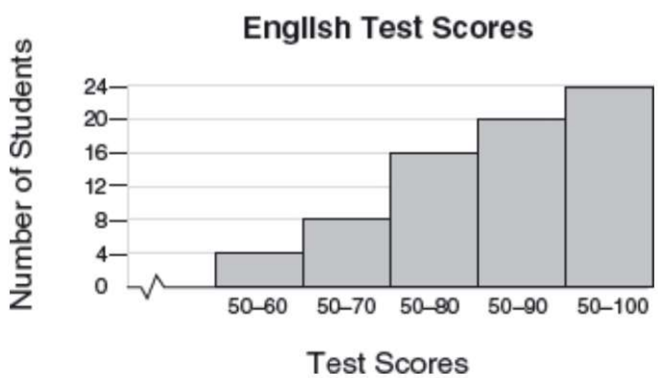
14. The greatest percent of increase in the jogger's heart rate occurred over what interval of time, in minutes?

(A) 0 to 1
(B) 1 to 2
(C) 2 to 3
(D) 3 to 4

HEAD CIRCUMFERENCE GROWTH SPEED	
Age	**Growth of Head Circumference (in centimeters)**
First year	$\text{Circumference} = \frac{\text{Height} + 12}{2}$
1 to 3 years	1 centimeter every 6 months
3 to 5 years	1 centimeter every year

15. The table above can be used to approximate the circumference of the head, in centimeters, during the first 5 years after birth. At 5 years of age, Jacob's head circumference was 81 cm. Based on the table, what was his approximate height, in centimeters, at 1 years old?

(A) 138
(B) 145
(C) 152
(D) 157



16. The cumulative histogram above shows the distribution of scores that 24 students received on an English test. If a student is selected at random, what is the probability that the student will have a score between 71 and 80?

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

	Tennis Team		
Gender	**Juniors**	**Seniors**	**Total**
Male	14	11	25
Female	5	10	15
Total	19	21	40

17. The table above shows the composition of a coed high school tennis team with a total of 40 members. A player who will be selected at random from the team will be given two free tickets to attend the U.S. Open Tennis Tournament. What is the probability that the tickets will be given to either a female junior player or a male senior player?

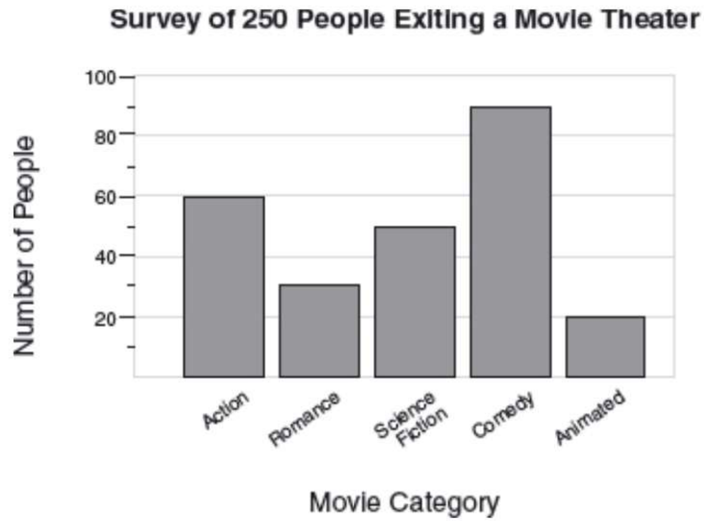
(A) $\frac{1}{8}$

(B) $\frac{1}{4}$

(C) $\frac{2}{5}$

(D) $\frac{1}{2}$

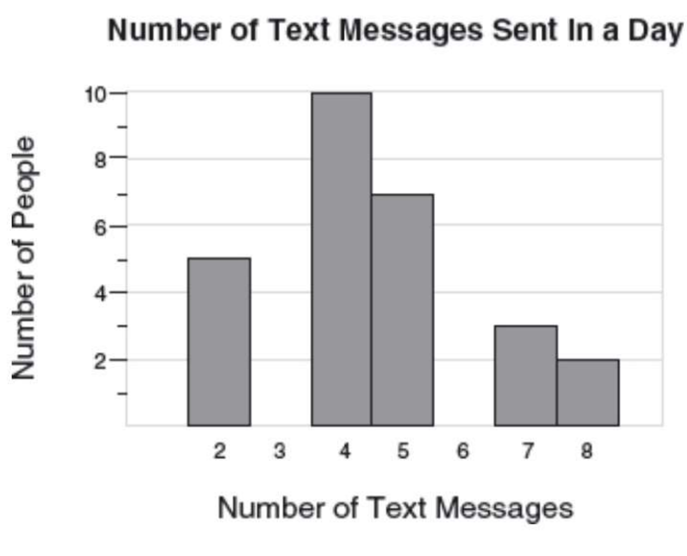
Grid-In



Questions 1 and 2 refer to the graph above that summarizes a survey of a group of 250 people who were randomly selected when leaving a multiplex movie theater and asked what type of movie they had seen.

1. What percent of the people surveyed saw either an action or a science fiction movie?

2. If a total of 1,700 tickets were sold for the five types of movies represented in the histogram, what is the best estimate for the number of tickets sold to the Romance movie?



3. The histogram above shows the number of mobile text messages sent by a randomly selected group of 27 people on a given day. The average

number of text messages sent per person is closest to what whole number?

Grade	1 Club	2 Clubs	3 or More Clubs	Total
9th	37	43	18	98
10th	48	38	22	108
11th	52	27	31	110
12th	75	30	29	134
Total	212	138	100	450

Questions 4–6 refer to the above table, which summarizes the results of a survey of the student body of a high school about club membership in which each student enrolled in the high school enumerated the clubs in which they were members.

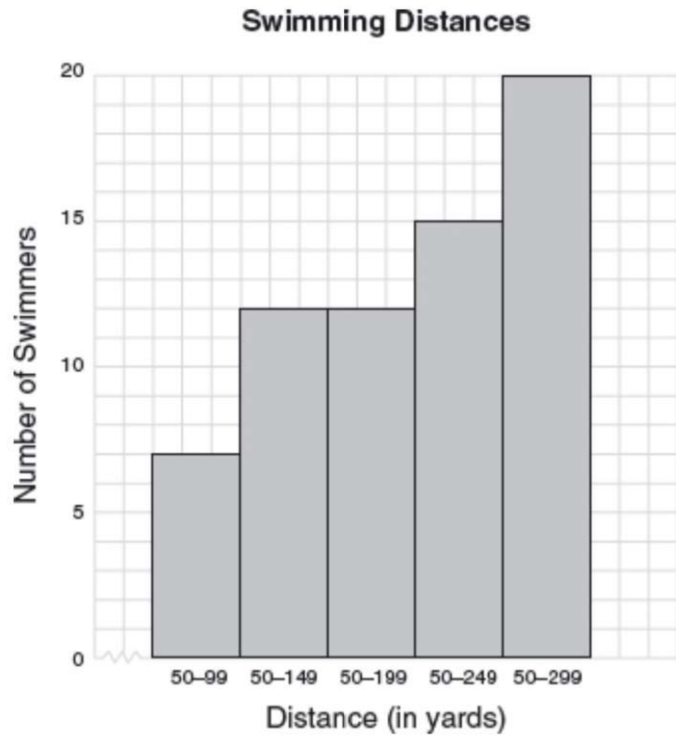
4. If a student is selected at random, what is the probability that the student does not belong to 3 or more clubs?

5. If 28% of the students who are enrolled in this school belong to at least two clubs, how many students who are enrolled in this school did *not* participate in the survey?

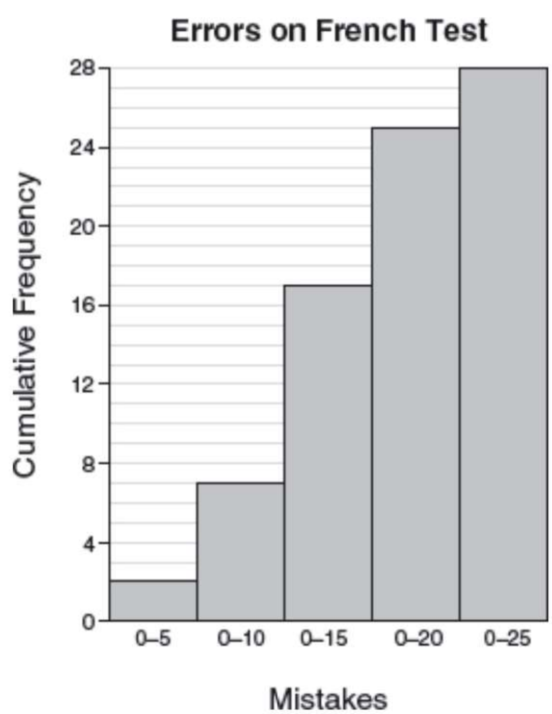
6. If a student is selected at random, what is the probability that the student is either a 9th grade student or belongs to 3 or more clubs?

	Eye Color			
Gender	**Brown**	**Hazel**	**Blue**	**Total**
Male		13		
Female		32		
Total	133			200

7. The partially completed table above describes the distribution of 200 subjects in a study involving eye color in which there were 4 times as many males with brown eyes as with blue eyes and 7 times as many females with brown eyes as with blue eyes. What percent of the subjects were either male with blue eyes or female with brown eyes?



8. Based on the cumulative histogram above, what percent of the total number of swimmers swam between 200 and 249 yards?



9. The cumulative histogram above shows the distribution of mistakes 28 students in a French language class made on a test. What is the

probability that a student selected at random made more than 10 mistakes?

LESSON 4-7 SCATTERPLOTS AND SAMPLING

OVERVIEW

A **scatterplot** is a graph that represents two sets of data as ordered pairs and shows them as points in the first quadrant of the *xy*-plane. The SAT may also include questions about sampling and errors associated with it. *Sampling* involves the selection of a smaller subset of individuals from within a larger statistical population for the purpose of making a generalization about the entire statistical population.

LINE OF BEST FIT

A scatterplot can indicate visually the type of relationship, if any, that exists between two sets of data. If the dots in a scatterplot are closely clustered about a line, as in Figure 4.8, then the relationship between the two sets of data can be approximated by a line. The **line of best fit** (or trend line) is the line that best represents the relationship between the two sets of data graphed in a scatterplot. Although there is a precise statistical method for determining the equation of the line of best fit, visually it is the line that can be drawn closest to all of the data points. Typically, the line of best fit passes through some, but not all of the plotted points with approximately the same number of data points falling on either side of it. A line of best fit is useful for predicting the y-value for any particular x-value that was not included in the original data.

Referring to Figure 4.8,

- Since the point (500, 560) lies on the line of best fit, a student with a verbal SAT score of 500 has a *predicted* math SAT score of 560.
- For only one data point, (400, **620**), does the math score differ by more than 100 points from the math score predicted by the line of best fit of (400, **480**).

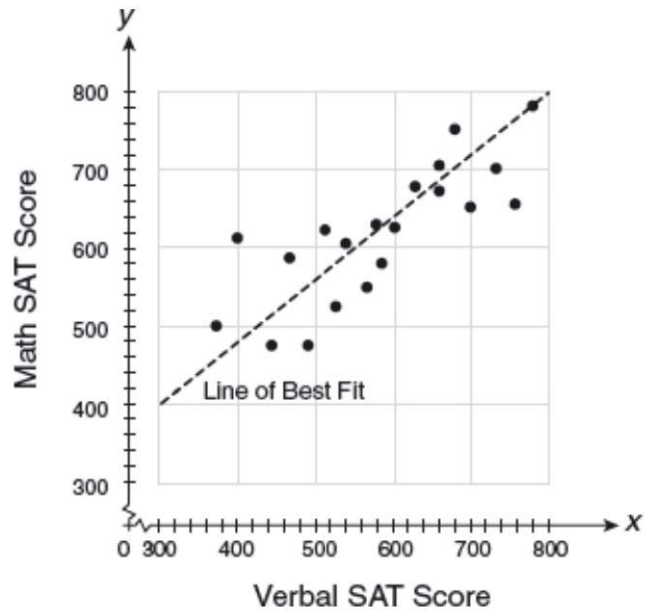
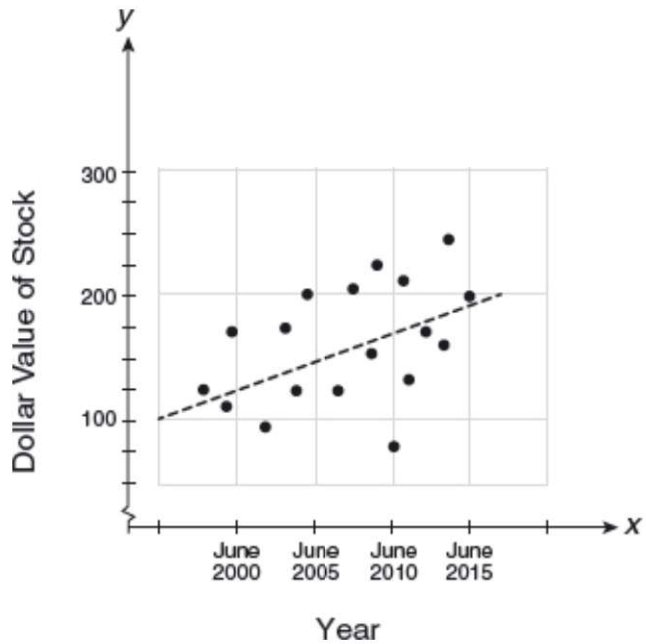


Figure 4.8 Scatterplot showing an approximately linear relationship between Verbal and Math SAT scores for students in the same SAT preparation class

MATH REFERENCE FACT

A scatterplot can show, at a glance, the relationship, if any, between two sets of data. It can show the type of relationship (linear, nonlinear, or no relationship), the strength of the relationship (as indicated by how closely the points fit a line or curve), and whether the data variables move in the same direction (if $x \uparrow$, $y \uparrow$) or in the opposite direction (if $x \uparrow$, $y \downarrow$).

➥ Example



The graph above shows how the value of a stock has increased over time. The line of best fit is shown.

a. The value of the stock increased from June 2000 to June 2005 by approximately what percent?

(A) 16%
(B) 20%
(C) 25%
(D) 28%

Solution

$$\begin{aligned}\text{Percent increase} &= \frac{\text{June 2005} - \text{June 2000}}{\text{June 2000}} \\ &= \frac{150-125}{125} \\ &= \frac{25}{125} \\ &= 20\%\end{aligned}$$

The correct choice is **(B)**.

b. What is the average yearly increase in value of the stock?

(A) $1
(B) $5
(C) $10
(D) $25

Solution

Find the slope of the line using the point (June 2000, 125) and (June 2005, 150):

$$\text{slope} = \frac{\$150 - \$125}{5 \text{ years}}$$
$$= \$5 \text{ per year}$$

The correct choice is **(B)**.

c. What is the greatest difference between the actual value of the stock and the value of the stock predicted by the line of best fit?

(A) $25
(B) $55
(C) $75
(D) $95

Solution

Greatest difference occurred in June 2010:

$$\text{Predicted value} - \text{Actual value} = \$170 - \$75 = \$95$$

The correct choice is **(D)**.

RECOGNIZING THE TYPE OF ASSOCIATION FROM A SCATTERPLOT

A scatterplot may suggest different types of relationships, or no relationship, between two sets of measurements.

□ Scatterplots may show a linear association between variables as in Figure 4.9.

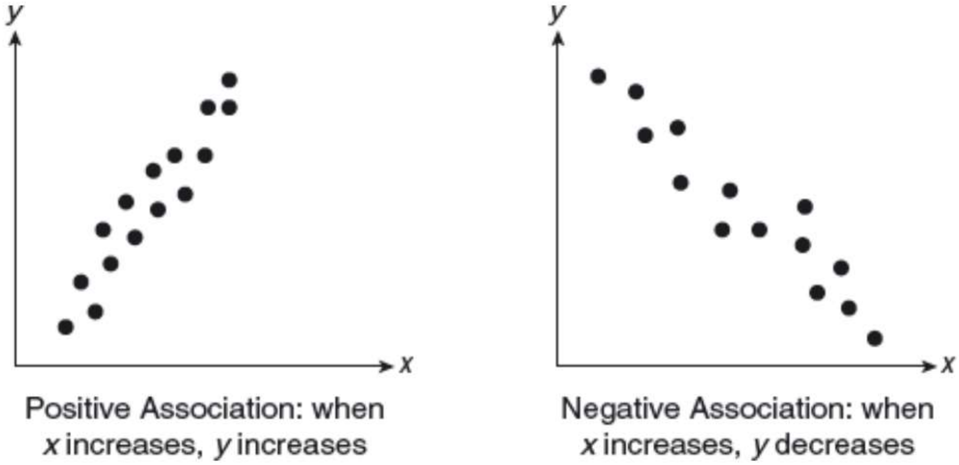


Figure 4.9 Comparing scatterplots with positive and negative linear associations

□ Scatterplots may show a nonlinear relationship between the data variables, as in Figure 4.10.

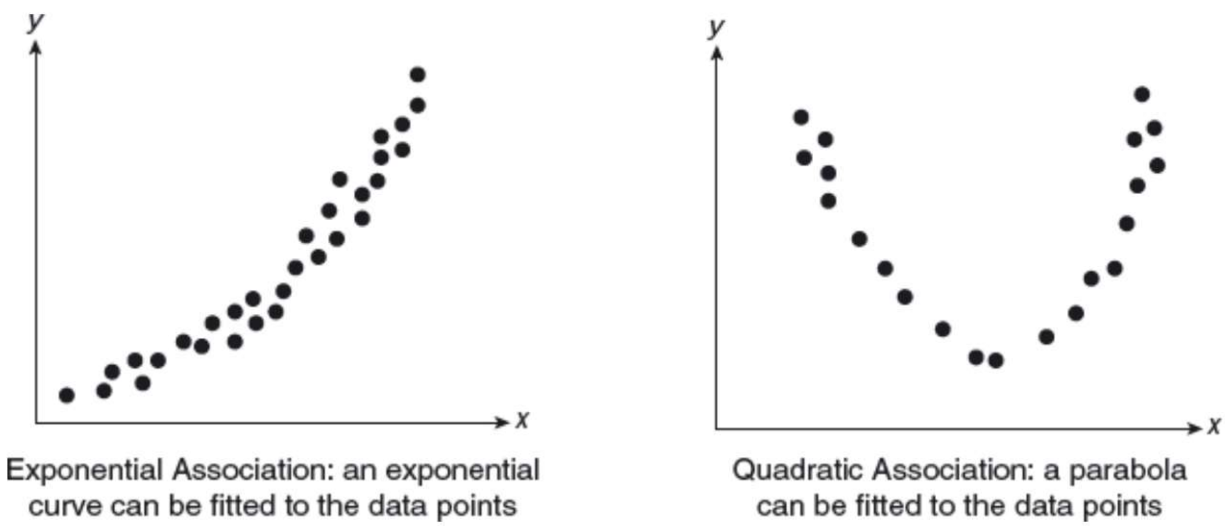


Figure 4.10 Comparing scatterplots with exponential and quadratic associations

TIP

Although a scatterplot may indicate two variables are related, it does NOT provide any information about cause and effect. For example, a scatterplot of children's shoe sizes plotted against their reading levels would show a positive association (since both are related to age), although a larger shoe size does not cause a higher reading level.

□ A scatterplot having no recognizable pattern indicates there is no relationship between the data variables, as illustrated in Figure 4.11.

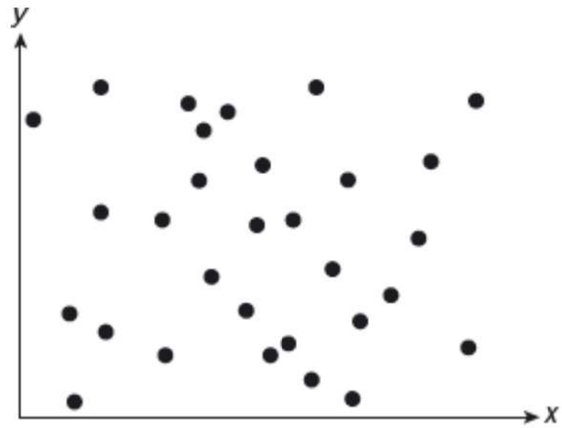


Figure 4.11 Scatterplot with no association between the variables

UNBIASED VERSUS BIASED SAMPLES

Suppose you wanted to conduct a survey to find out how students in your school feel about an issue that affects the entire student body. It may not be practical or even possible to interview every student. In such cases, a smaller but representative group of students can be selected to be interviewed using some random selection process such as choosing one out of every five students entering the school cafeteria on a regular school day. The entire school population represents the target **population** and the smaller subset of students selected to be interviewed is the **sample**. A sample may be unbiased or biased.

- A sample is **unbiased** if each member of the target population has an equal chance of being selected for the sample.

- If the selection process is not random and favors a particular group within the target population, then the sample is **biased**.

Because the students selected to participate in this survey are chosen at random from the entire student body, the sample is *unbiased*. If only senior boys who are members of an athletic team were selected to participate in the survey, then the sample would be biased. Any general conclusions drawn from a biased survey may not be valid.

MARGIN OF ERROR

The **margin of error** for a survey based on a sample is a statistic that estimates the extent to which the conclusions drawn may differ from the results that would have been obtained from surveying the entire statistical population. In general, the larger and more random the sample, the smaller the margin of error.

> **TIP**
>
> **Experimental or laboratory data may also be subject to a margin of error due to inaccuracies in observation and measurement. The SAT may ask questions about situations that involve a margin of error, but it will *not* ask you to calculate it.**

CONFIDENCE INTERVALS

Anytime a sample is collected and a statistic such as the mean of the sample is calculated from it, the question arises regarding how accurately the sample mean represents the true population mean. A **confidence interval** for a sample statistic such as the mean is a range of values for which we expect the true population mean to lie within. Associated with a confidence interval is a probability value that reflects the degree of certainty that confidence intervals drawn from the same population will contain the actual population statistic. A "95% confidence interval for the mean" expresses the idea that if repeated samples were drawn from the same target population and a 95% confidence interval calculated for each sample, then 95% of those samples will contain the population mean. To illustrate this idea, suppose at the service center of a car dealership 65% of the people surveyed reported that they expect to buy or lease a new car within the next 3 years. A statistician calculates the confidence level to be 95% for an interval of 5% below and above the 65% mark. This means that if the same survey were to be repeated 100 times, we could assume that the percentage of people who would report that they expect to buy or lease a new car within the next 3 years would range between 60% and 70% in 95 of the 100 surveys.

When discussing confidence intervals, it is important to remember that:

- Probability values other than "95%" can be used.
- Other sample statistics such as standard deviation can be used.
- Reducing the width of the confidence interval, produces a more accurate estimate of the actual population statistic. This can often be accomplished by increasing the sample size and by reducing the variability of the data. Reducing the variability of the data and, as a result, reducing the standard deviation, may require improving the accuracy of numerical data measurements or modifying the design of the survey.

LESSON 4-7 TUNE-UP EXERCISES

Multiple-Choice

1. Which survey is most likely to have the *least* bias?

   (A) surveying a sample of people leaving a movie theater to determine which flavor of ice cream is the most popular
   (B) surveying the members of a football team to determine the most watched TV sport
   (C) surveying a sample of people leaving a library to determine the average number of books a person reads in a year
   (D) surveying a sample of people leaving a gym to determine the average number of hours a person exercises per week

2. Erica is conducting a survey about the proposed increase in the sports budget in the Hometown School District. Which survey method would likely contain the *most* bias?

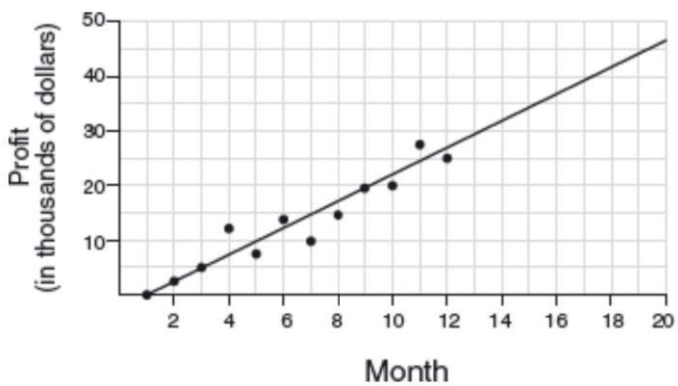
   (A) Erica asks every third person entering the Hometown Grocery Store
   (B) Erica asks every third person leaving the Hometown Shopping Mall this weekend
   (C) Erica asks every fifth student entering Hometown High School on Monday morning
   (D) Erica asks every fifth person leaving Saturday's Hometown High School football game

Ages of People in Survey on Driving Habits

Age Group	Number of Drivers
16–25	150
26–35	129
36–45	33
46–55	57
56–65	31

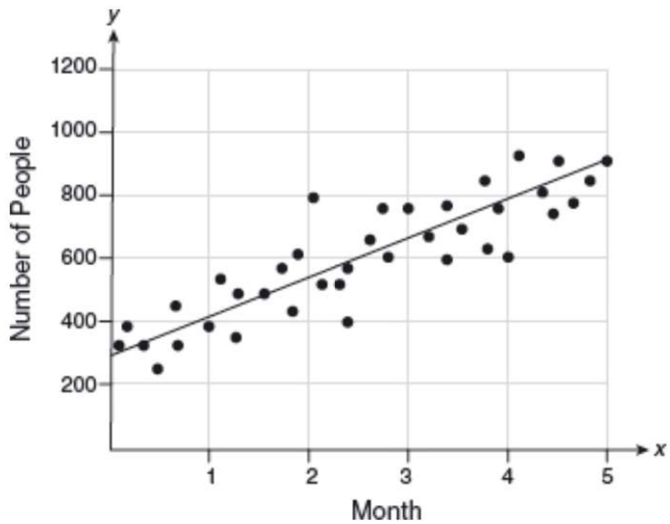
3. The table above summarizes the number of people by age group who were included in a survey of driving habits. Which of the following statements is true?

(A) The survey was not biased since different age groups were included.
(B) The survey was biased because individuals 36 and older were underrepresented.
(C) The survey was biased because it did not differentiate between males and females.
(D) The survey was not biased since a large number of drivers were polled.



4. The scatterplot above shows the profit, by month, for a new company for the first year of operation. A line of best fit is also shown. Using this line, by what dollar amount did the profit in the 18th month exceed the profit in the 13th month?

(A) \$5,000
(B) \$7,750
(C) \$12,500
(D) \$15,000



Questions 5–7 refer to the scatterplot above.

A new fitness class was started at several fitness clubs owned by the same company. The scatterplot shows the total number of people attending the class during the first five months in which the class was offered. The line of best fit is drawn.

5. For month 4, the predicted number of people attending the class was approximately what percent greater than the actual number of people attending the class?

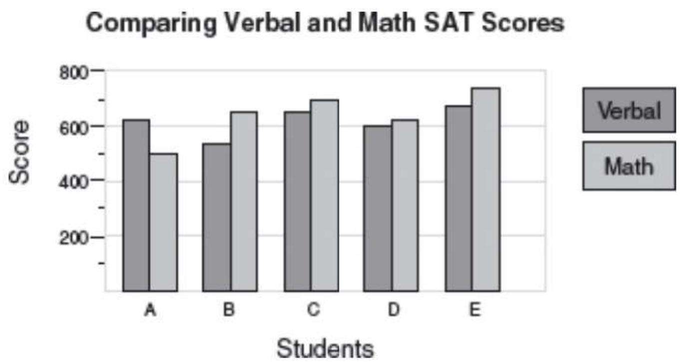
   (A) 15%
   (B) 20%
   (C) 30%
   (D) 36%

6. During the five-month period, the average increase in the number of people attending the class per month is closest to which of the following?

   (A) 80
   (B) 100
   (C) 120
   (D) 140

7. At the beginning of which month did the actual number of people attending the class differ by the greatest amount from the number predicted by the line of best fit?

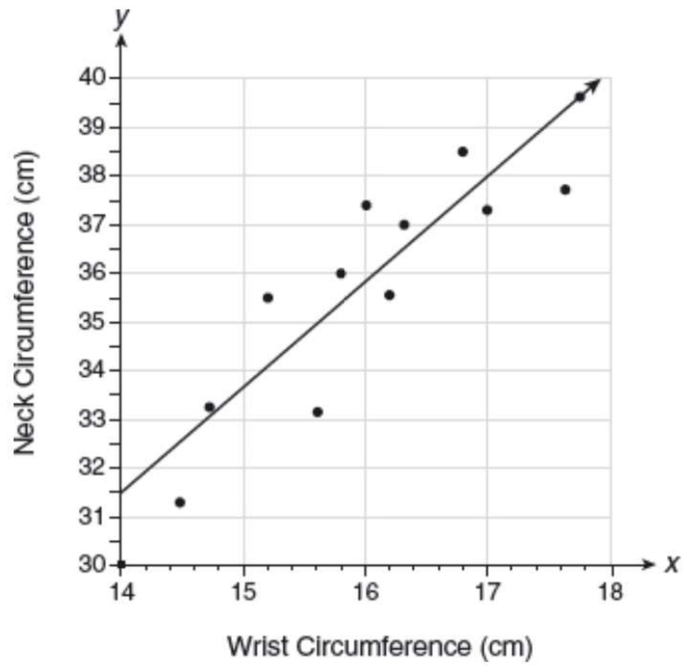
(A) month 2
(B) month 3
(C) month 4
(D) month 5



8. The bar graph above shows the verbal and math SAT scores for five students labeled A through E. If a scatterplot of the data in the bar graph is made such that the math SAT score for each student is plotted along the x-axis and their verbal SAT score is plotted along the y-axis, how many of the data points would lie above the line $y = x$?

(A) 1
(B) 2
(C) 3
(D) 4

Grid-In



Questions 1–3 refer to the above scatterplot, which shows wrist and neck circumference measurements, in centimeters, for 12 people. The line of best fit is drawn.

1. What is the predicted neck circumference, in centimeters, for someone whose wrist circumference is 17.0 cm?

2. How many of the 12 people have an actual neck circumference that differs by more than 1 centimeter from the neck circumference predicted by the line of best fit?

3. What is the average increase in neck circumference per centimeter increase in wrist circumference, correct to the *nearest tenth*?

LESSON 4-8 SUMMARIZING DATA USING STATISTICS

OVERVIEW

A **statistic** is a single number used to describe a set of data values. The *mean*, *median*, and *mode* are statistics that measure *central tendency* by describing the central location of the data within the set. The *range* and *standard deviation* are statistics that describe the extent to which the data are spread out. The SAT may ask you to calculate the mean, median, mode, and range, but not the standard deviation. You may be asked a question, however, that asks you to compare the standard deviations of two sets of data based on how spread out the individual data values in each set appear to be from the means.

FINDING THE AVERAGE (ARITHMETIC MEAN)

To find the average of a set of *n* numbers, add all the numbers together and then divide the sum by *n*. Thus,

$$\text{Average} = \frac{\text{Sum of the } n \text{ values}}{n}$$

For example, the average of three exam grades of 70, 80, and 72 is 74 since

$$\text{Average} = \frac{70 + 80 + 72}{3} = \frac{222}{3} = 74$$

FINDING AN UNKNOWN NUMBER WHEN AN AVERAGE IS GIVEN

If you know the average of a set of *n* numbers, you can find the sum of the *n* values by using this relationship:

$$\text{Sum of the } n \text{ values} = \text{Average} \times n$$

➡ Example

The average of a set of four numbers is 78. If three of the numbers in the set are 71, 74, and 83, what is the fourth number?

Solution

If the average of four numbers is 78, the sum of the four numbers is $78 \times 4 = 312$. The sum of the three given numbers is $71 + 74 + 83 = 228$. Since $312 - 228 = 84$, the fourth number is **84**.

➡ Example

The average of w, x, y, and z is 31. If the average of w and y is 24, what is the average of x and z?

Solution 1

- Since $\frac{w+x+y+z}{4} = 31$, $w + x + y + z = 4 \times 31 = 124$.
- Since $\frac{w+y}{2} = 24$, $w + y = 2 \times 24 = 48$.
- Thus, $x + z + 48 = 124$, so $x + z = 76$.
- Since $x + z = 76$, $\frac{x+z}{2} = \frac{76}{2} = 38$.

Hence, the average of x and z is **38**.

Solution 2

Since the average of two of the four numbers is 7 ($= 31 - 24$) less than the average of the four numbers, the average of the other two numbers must be 7 more than the average of the four numbers. Hence, the average of x and z is $31 + 7 = 38$.

FINDING THE WEIGHTED AVERAGE

A **weighted average** is the average of two or more sets of numbers that do not contain the same number of values. To find the weighted average of two or more sets of numbers, proceed as follows:

- Multiply the average of each set by the number of values in that set, and then add the products.
- Divide the sum of the products by the total number of values in all of the sets.

➥ Example

In a class, 18 students had an average midterm exam grade of 85 and the 12 remaining students had an average midterm exam grade of 90. What is the average midterm exam grade of the entire class?

Solution

$$
\begin{aligned}
\text{Weighted average} &= \frac{\overbrace{(\text{Average} \times \text{Number})}^{\text{Group 1}} + \overbrace{(\text{Average} \times \text{Number})}^{\text{Group 2}}}{\text{Total number of students}} \\
&= \frac{(85 \times 18) + (90 \times 12)}{30} \\
&= \frac{1{,}530 \quad + 1{,}080}{30} \\
&= \frac{2{,}610}{30} \\
&= 87
\end{aligned}
$$

FINDING THE MEDIAN

To find the median of a set of numbers, first arrange the numbers in size order.

- If a set contains an odd number of values, the median is the middle value. For example, the median of the set of numbers

  $$8, 12, 15, \underbrace{17}_{\text{Median}}, 19, 20, 25$$

  is 17 since the number of values in the set that are less than 17 is the same as the number of values in the set that are greater than 17.
- If a set contains an even number of values, the median is the average (arithmetic mean) of the two middle values. For example, the set of

numbers

$$10, 20, 24, 30, 40, 50$$

contains six values. Since the two middle values in the set are 24 and 30, the median is their average:

$$10, 20, \underbrace{24, 30}, 40, 50 \quad \text{Median} = \frac{24+30}{2} = 27$$

FINDING THE MODE

The mode of the set

$$7, 2, 6, 3, 2, 6, 7, 3, 9, 6$$

is 6, since 6 appears more times than any other number in the set.

RANGE AND STANDARD DEVIATION

The *range* and *standard deviation* provide information about how spread out the data are.

- The **range** is the difference between the greatest and smallest data values.
- The **standard deviation** is a statistic that measures how far apart the individual data scores are from the mean. A set of data scores in which the data values are clustered around the mean will have a smaller standard deviation than a data set in which the individual data values are more spread out and further from the mean.

TRANSFORMING AN ENTIRE SET OF DATA

If each score in a set of data values is changed in the same way, you can predict how the different statistical measures are affected without recalculating these statistics.

- When each value in a set of data is increased (or decreased) by the same nonzero number, then the mean, median, and mode are each increased (or decreased) by that number. The *range*, which is the difference between the highest and lowest values in the data set, is *not* affected. If the mean of a set of test scores is 78 and *each* test score is increased by 6, then the mean of the revised set of test scores is $78 + 6 = 84$.
- When each value in a set of data is multiplied (or divided) by the same nonzero number, then the mean, median, mode, and range are each multiplied (or divided) by that number. Suppose the mean number of music CDs that five friends own is 13. Mary predicts that six months from now each of the five friends will own twice as many music CDs as they do now. Based on Mary's prediction, the mean number of music CDs they will own is $13 \times 2 = 26$.

LESSON 4-8 TUNE-UP EXERCISES

Multiple-Choice

1. The average (arithmetic mean) of a set of seven numbers is 81. If one of the numbers is discarded, the average of the remaining numbers is 78. What is the value of the number that was discarded?

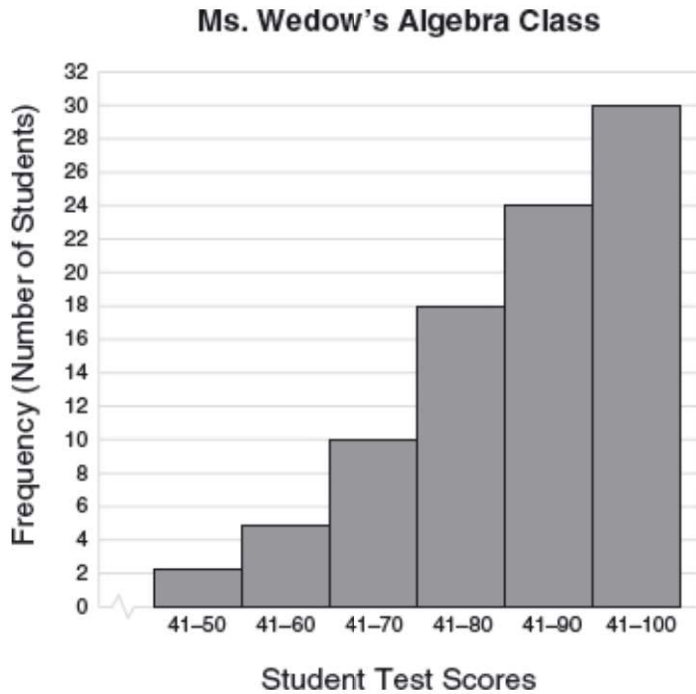
   (A) 98
   (B) 99
   (C) 100
   (D) 101

2. The arithmetic mean of a set of 20 test scores is represented by x. If each score is increased by y points, which expression represents the arithmetic mean of the revised set of test scores?

   (A) $x + y$
   (B) $x + 20y$
   (C) $x + \frac{y}{20}$
   (D) $\frac{x + y}{20}$

3. What is the area of the circle whose radius is the average of the radii of two circles with areas of 16π and 100π?

   (A) 25π
   (B) 36π
   (C) 49π
   (D) 64π



4. The diagram above shows a graph of the students' test scores in Ms. Wedow's algebra class. Which *ten-point* interval contains the median?

(A) 61–70
(B) 71–80
(C) 81–90
(D) 91–100

5. If k is a positive integer, which of the following represents the average of 3^k and 3^{k+2}?

(A) $\frac{1}{2} \cdot 3^{k+1}$
(B) $5 \cdot 3^k$
(C) $6^{\frac{3}{2}k}$
(D) $\frac{1}{2} \cdot 3^{3k}$

6. When x is subtracted from $2y$, the difference is equal to the average of x and y. What is the value of $\frac{x}{y}$?

(A) $\frac{1}{2}$
(B) $\frac{2}{3}$

(C) 1
(D) $\frac{3}{2}$

7. If the average of x, y, and z is 32 and the average of y and z is 27, what is the average of x and $2x$?

(A) 42
(B) 45
(C) 48
(D) 63

Company 1

Worker's Age in Years	Salary in Dollars
25	30,000
27	32,000
28	35,000
33	38,000

Company 2

Worker's Age in Years	Salary in Dollars
25	29,000
28	35,500
29	37,000
31	65,000

8. Which of the following statements is true about the data in the tables above?

I. The mean salaries for both companies are greater than $35,000.
II. The mean age of workers in Company 1 is greater than the mean age of workers in Company 2.
III. The salary range in Company 2 is greater than the salary range in Company 1.

(A) I only
(B) III only
(C) I and II only
(D) II and III only

9. A man drove a car at an average rate of speed of 45 miles per hour for the first 3 hours of a 7-hour car trip. If the average rate of speed for the entire trip was 53 miles per hour, what was the average rate of speed in miles per hour for the remaining part of the trip?

(A) 50

(B) 55
(C) 57
(D) 59

10. In a set of n data values, m represents the median. If each number in the set is decreased by 3, which expression represents the median of the revised set of data values?

(A) m
(B) $m - 3$
(C) $m - \frac{3}{n}$
(D) $\frac{m-3}{n}$

11. Susan received grades of 78, 93, 82, and 76 on four math exams. What is the lowest score she can receive on her next math exam and have an average of at least 85 on the five exams?

(A) 96
(B) 94
(C) 92
(D) 90

12. What is the average of $(x + y)^2$ and $(x - y)^2$?

(A) $\frac{x+y}{2}$
(B) xy
(C) $x^2 - y^2$
(D) $x^2 + y^2$

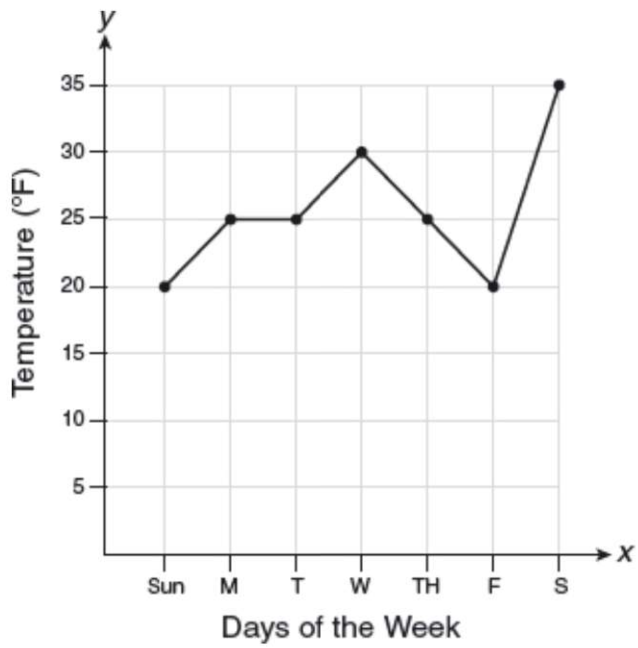
13. The average of the test scores of a group of x students is 76 and the average of the test scores of a group of y students is 90. When the scores of the two groups of students are combined, the average test score is 85. What is the value of $\frac{x}{y}$?

(A) $\frac{4}{7}$

(B) $\frac{5}{9}$

(C) $\frac{2}{3}$

(D) $\frac{7}{4}$



14. The graph above shows the average daily temperature during a particular week in January in a certain city. Which statement best describes the temperature data in the graph above?

(A) median = mean
(B) mean < mode
(C) median = mode
(D) mean = mode

15. The average of a, b, c, d, and e is 28. If the average of a, c, and e is 24, what is the average of b and d?

(A) 31
(B) 32
(C) 33
(D) 34

16. If $2a + b = 7$ and $b + 2c = 23$, what is the average of a, b, and c?

(A) 5
(B) 7.5
(C) 15
(D) 12.25

Minutes	14	15	16	17	18	19	20
Number of Students	5	3	x	5	2	10	1

17. The number of minutes students took to complete a quiz is summarized in the table above. If the mean number of minutes was 17, which equation could be used to calculate x?

(A) $17 = \frac{119 + x}{x}$

(B) $17 = \frac{119 + 16x}{x}$

(C) $17 = \frac{446 + x}{26 + x}$

(D) $17 = \frac{446 + 16x}{26 + x}$

18. The average of a, b, c, and d is p. If the average of a and c is q, what is the average of b and d in terms of p and q?

(A) $2p + q$
(B) $2p - q$
(C) $2q + p$
(D) $2q - p$

19. The lowest value in a set of ordered scores is x and the highest value is y. If each score is increased by k, then which of the following must be true of the revised set of scores?

I. The mean is increased by k.
II. The range is k.
III. The median remains unchanged.

(A) I only
(B) II only

(C) I and III
(D) II and III

Class *X*

Grade	Frequency
A	4
B	11
C	3
D	2
F	1

Class *Y*

Grade	Frequency
A	6
B	4
C	2
D	6
F	3

20. The tables above give the distribution of grades for 21 students in two different college mathematics classes. For purposes of making statistical calculations, A = 4, B = 3, C = 2, D = 1, and F = 0. Which of the following statements is true about the data shown for these two classes?

I. The standard deviation of grades is greater for class *X*.
II. The standard deviation of grades is greater for class *Y*.
III. The median letter grade is the same for classes X and Y.

(A) I only
(B) II only
(C) I and III only
(D) II and III only

Player's Annual Salaries (millions of dollars)					
0.5	0.5	0.6	0.7	0.75	0.8
1.0	1.0	1.1	1.25	1.3	1.4
1.4	1.8	2.5	3.7	3.8	4
4.2	4.6	5.1	6	6.3	7.2

21–22 The table above shows the annual salaries for the 24 members of a professional sports team in terms of millions of dollars.

21. If each player's salary is increased by 10%, which of the following statistics does *not* increase by 10%?

(A) median
(B) mean

(C) mode
(D) all increase by 10%

22. The team signs an additional player to a contract with an annual salary of 7.5 million dollars per year, which brings the sum of the salaries of the 25 players to 69 million dollars. By what amount, in dollars, does the mean increase?

(A) 197,500
(B) 256,250
(C) 300,000
(D) It cannot be determined.

Grid-In

1. The average of r and s is 7.5, and the average of r, s and t is 11. What is the value of t?

2. If the average of x, y, and z is 12, what is the average of $3x$, $3y$, and $3z$?

3. In order to compensate for a difficult midterm exam, Danielle's mathematics teacher adjusted each of the 25 students' midterm exam scores by replacing it by one-half of the original score increased by 50. If the mean of the revised set of midterm scores is 82, what was the mean of the original set of scores?

4. On a test that has a normal distribution of scores, a score of 58 falls two standard deviations below the mean, and a score of 85 is one standard deviation above the mean. What is the mean score of this test?

5

Passport to Advanced Math

This chapter extends the facts, concepts, and skills covered in the Heart of Algebra chapter. Particular emphasis is placed on analyzing and manipulating more complicated algebraic expressions, equations, and functions. "Passport to Advanced Math" represents the third of the four major mathematics content groups tested by the redesigned SAT.

LESSONS IN THIS CHAPTER

Lesson 5-1 Rational Exponents
Lesson 5-2 More Advanced Algebraic Methods
Lesson 5-3 Complex Numbers
Lesson 5-4 Completing the Square
Lesson 5-5 The Parabola and Its Equations
Lesson 5-6 Reflecting and Translating Function Graphs

LESSON 5-1 RATIONAL EXPONENTS

OVERVIEW

In order to give meaning to expressions such as 4^{-1} or 4^0 or $4^{\frac{1}{2}}$, the rules for exponents can be extended to include exponents that are rational numbers. Equations may contain variables with rational exponents or variables underneath a radical sign.

ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

The rules for working with zero, negative, and fractional exponents are summarized in Table 5.1.

Table 5.1 Rules for Rational Exponents

Zero Exponent Rule ($x \neq 0$)	Negarive Exponent Rule ($x \neq 0$)	Fractional Exponent Rule ($x \geq 0$ when n is even)
$x^0 = 1$	• $x^{-a} = \frac{1}{x^a}$ • $\frac{1}{x^{-a}} = x^a$	• $\frac{1}{x^n} = \sqrt[n]{x}$ • $x^{\frac{a}{n}} = \sqrt[n]{x^a} = \left(\sqrt[n]{x}\right)^a$
EXAMPLE: $(2x)^0 = 1$	EXAMPLE: $\frac{a^{-4}}{b^{-2}c^5} = \frac{b^2}{a^4c^5}$	EXAMPLE: $(-27)^{\frac{2}{3}} = \left(\sqrt[3]{-27}\right)^2 = (-3)^2 = 9$

MATH REFERENCE FACT

$x^{\frac{1}{2}} = \sqrt{x}$, provided $x \geq 0$

Here are some examples:

- $3x^{-2} = \frac{3}{x^2}$
- $\frac{\sqrt[3]{x}}{\sqrt{x}} = \frac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}} = x^{-\frac{1}{6}}$ or $\frac{1}{x^{\frac{1}{6}}}$
- $\frac{bc^{-5}}{2b^{-3}} = \frac{b \cdot b^3}{2c^5} = \frac{b^4}{2c^5}$
- $\frac{x^5y^2}{(xy^3)^2} = \frac{x^5y^2}{x^2y^6} = \frac{x^3}{y^4}$ or x^3y^{-4}
- $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = (2)^4 = 16$
- $\frac{x^2}{\sqrt[3]{x}} = \frac{x^2}{x^{\frac{1}{3}}} = x^{2-\frac{1}{3}} = x^{\frac{5}{3}}$ or $\sqrt[3]{x^5}$ or $\left(\sqrt[3]{x}\right)^5$

➡ Example

If $x > 1$ and $\frac{\sqrt{x^3}}{x^2}$, what is the value of n?

(A) $-\frac{3}{2}$

(B) -1

(C) $-\frac{1}{2}$

(D) $\frac{1}{2}$

Solution

- Rewrite the radical using the exponent rule for equare roots:

$$\frac{\sqrt{x^3}}{x^2} = x^n$$

$$\frac{(x^3)^{\frac{1}{2}}}{x^2} = x^n$$

$$\frac{x^{\frac{3}{2}}}{x^2} = x^n$$

- Use the quotient law of exponents:

$$x^{\frac{3}{2}-2} = x^n$$

■ Simplify the exponent:

$$x^{\frac{3}{2}-\frac{4}{2}} = x^n$$
$$x^{-\frac{1}{2}} = x^n$$
$$n = -\frac{1}{2}$$

The correct choice is **(C)**.

RADICAL EQUATIONS

If an equation contains the square root of a variable term, isolate the radical in the usual way. Eliminate the radical by raising both sides of the equation to the second power. To solve $\sqrt{3x+1} - 2 = 3$ perform the following steps:

1. Isolate the radical:

$$\sqrt{3x+1} = 5$$

2. Raise both sides of the equation to the second power:

$$\left(\sqrt{3x+1}\right)^2 = (5)^2$$

3. Simplify:

$$3x + 1 = 25$$
$$\frac{3x}{3} = \frac{24}{3}$$
$$x = 8$$

MATH REFERENCE FACT

To solve an equation of the form $\sqrt[n]{x}$, raise both sides of the equation to the *n*th power:

$$\left(\sqrt[n]{x}\right)^n = (c)^n \text{ so } x = c^n$$

EQUATIONS WITH FRACTIONAL EXPONENTS

Since the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$, raising both sides of the equation $x^{\frac{3}{2}} = 8$ to the $\frac{2}{3}$ power creates an equivalent equation in which the power of x is 1:

$$x^{\frac{3}{2}\cdot\frac{2}{3}} = (8)^{\frac{2}{3}}$$
$$x^1 = \left(\sqrt[3]{8}\right)^2$$
$$x = (2)^2 = 4$$

MATH REFERENCE FACT

To solve an equation of the form $x^{\frac{m}{n}} = k$, raise both sides of the equation to the $\frac{n}{m}$ power so that the exponent of x becomes 1.

EXPONENTIAL EQUATIONS

If an equation contains a variable in an exponent, rewrite each side as a power of the same base. Then set the exponents equal to each other. For example, solve $8^{x-2} = 4^{x+1}$ as follows:

1. Rewrite each side as a power of 2: $(2^3)^{(x-2)} = (2^2)^{(x+1)}$
2. Simplify the exponents: $2^{3(x-2)} = 2^{2(x+1)}$
3. Set the exponents equal: $3(x-2) = 2(x+1)$

$$3x - 6 = 2x + 2$$
$$3x - 2x = 2 + 6$$
$$x = 8$$

LESSON 5-1 TUNE-UP EXERCISES

Multiple-Choice

1. Which of the following is equal to $b^{-\frac{1}{2}}$ for all values of b for which the expression is defined?

   (A) $\frac{b}{b^2}$

   (B) $\frac{\sqrt{b}}{b}$

   (C) $\frac{1}{\sqrt{2b}}$

   (D) $\frac{1}{2}b$

2. Which expression is equivalent to $(9x^2y^6)^{-\frac{1}{2}}$?

   (A) $\frac{1}{3xy^3}$

   (B) $3xy^3$

   (C) $\frac{3}{xy^3}$

   (D) $\frac{xy^3}{3}$

3. If $4^y + 4^y + 4^y + 4^y = 16^x$, then $y =$

   (A) $2x - 1$
   (B) $2x + 1$
   (C) $x - 2$
   (D) $x + 2$

4. If $\sqrt{m} = 2p$, then $m^{\frac{3}{2}} =$

   (A) $\frac{p}{3}$
   (B) $2p^2$
   (C) $6p^3$
   (D) $8p^3$

5. If $3^x = 81$ and $2^{x+y} = 64$, then $\frac{x}{y} =$

(A) 1

(B) $\frac{3}{2}$

(C) 2

(D) $\frac{5}{2}$

6. Which of the following is equal to $y^{\frac{3}{2}}$ for all values of y for which the expression is defined?

(A) $\sqrt[3]{y^2}$

(B) $\sqrt{y^3}$

(C) $\sqrt[3]{y^{\frac{1}{2}}}$

(D) $3\sqrt{y}$

7. Which expression is equivalent to $\frac{(2xy)^{-2}}{4y^{-5}}$?

(A) $-\frac{y^3}{x^2}$

(B) $-\frac{y^3}{16x^2}$

(C) $\frac{y^3}{x^2}$

(D) $\frac{y^3}{16x^2}$

8. If $10^k = 64$, what is the value of $10^{\frac{k}{2}+1}$?

(A) 18
(B) 42
(C) 80
(D) 81

9. If x is a positive integer greater than 1, how much greater than x^2 is $x^{\frac{5}{2}}$?

(A) $x^2\left(1 - x^{\frac{1}{2}}\right)$

(B) $x^{-\frac{1}{2}}$

(C) $x^2\left(x^{\frac{1}{2}} - 1\right)$

(D) $x^{\frac{1}{2}}$

10. The expression $\frac{x^2}{\sqrt{x^3}}$ is equivalent to

(A) $\sqrt[3]{x}$

(B) $\frac{1}{\sqrt{x}}$

(C) $\sqrt{x}$

(D) $\frac{1}{\sqrt[3]{x^2}}$

11. If n and p are positive integers such that $8(2^p) = 4^n$, what is n in terms of p?

(A) $\frac{p+2}{3}$

(B) $\frac{2p}{3}$

(C) $\frac{p+3}{2}$

(D) $\frac{3p}{2}$

$$2\sqrt{x-k} = x-6$$

12. If $k = 3$, what is the solution of the equation above?

(A) $\{4, 12\}$
(B) $\{3\}$
(C) $\{4\}$
(D) $\{12\}$

13. When $x^{-1} - 1$ is divided by $x - 1$, the quotient is

(A) -1

(B) $-\frac{1}{x}$

(C) $\frac{1}{x^2}$

(D) $\frac{1}{(x-1)^2}$

14. If n is a negative integer, which statement is *always* true?

(A) $6n^{-2} < 4n^{-1}$

(B) $\frac{n}{4} > -6n^{-1}$

(C) $6n^{-1} < 4n^{-1}$

(D) $4n^{-1} > (6n)^{-1}$

$$g(x) = a\sqrt{a(1-x)}$$

15. Function g is defined by the equation above. If $g(-8) = 375$, what is the value of a?

(A) 25
(B) 75
(C) 125
(D) 625

16. If $27^x = 9^{y-1}$, then

(A) $y = \frac{3}{2}x + 1$

(B) $y = \frac{3}{2}x + 2$

(C) $y = \frac{3}{2}x + \frac{1}{2}$

(D) $y = \frac{1}{2}x + \frac{2}{3}$

Grid-In

$$\sqrt{3p^2 - 11} - x = 0$$

1. If $p > 0$ and $x = 8$ in the equation above, what is the value of p?

2. If $x^{-\frac{1}{2}} = \frac{1}{8}$, what is the value of $x^{\frac{2}{3}}$?

3. If y is not equal to 0, what is the value of $\frac{6(2y)^{-2}}{(3y)^{-2}}$?

4. If $(2rs)^{-1} = 3s^{-2}$, what is the value of $\frac{r}{s}$?

5. If m and p are positive integers and $(2\sqrt{2})^m = 32^p$, what is the value of $\frac{p}{m}$?

6. If a, b, and c are positive numbers such that $\sqrt{\frac{a}{b}} = 8c$ and $ac = b$, what is the value of c?

7. If $k = 8\sqrt{2}$ and $\frac{1}{2}k = \sqrt{3h}$, what is the value of h?

8. If $64^{2n+1} = 16^{4n-1}$, what is the value of n?

$$\frac{\sqrt[3]{a^8}}{\left(\sqrt{a}\right)^3} = a^x, \text{ where } a > 1$$

9. In the equation above, what is the value of x?

10. A meteorologist estimates how long a passing storm will last by using the function $t(d) = 0.08d^{\frac{3}{2}}$, where d is the diameter of the storm, in miles, and t is the time, in hours. If the storm lasts 16.2 minutes, find its diameter, in miles.

LESSON 5-2 MORE ADVANCED ALGEBRAIC METHODS

OVERVIEW

This section extends some previously covered algebraic skills related to factors, equations, and more complicated algebraic expressions.

FACTORING BY GROUPING PAIRS OF TERMS

Some polynomials with four terms can be factored by grouping appropriate pairs of terms together and then factoring out a binomial that is common to both pairs of terms.

- $$\begin{aligned} ab - 4b + 3a - 12 &= (ab - 4b) + (3a - 12) \\ &= b\underline{\underline{(a-4)}} + 3\underline{\underline{(a-4)}} \\ &= (a-4)(b+3) \end{aligned}$$
- $$\begin{aligned} x^3 + 5x^2 - 9x - 45 &= x^2\underline{\underline{(x+5)}} - 9\underline{\underline{(x+5)}} \\ &= (x+5)(x^2-9) \\ &= (x+5)(x+3)(x-3) \end{aligned}$$

ZEROS OF A FUNCTION

A **zero** of a function is any value of the variable that makes the function evaluate to zero. The zeros of a function correspond to the *x*-intercepts of the graph of the function.

➥ Example

$$f(x) = 2x^3 - 5x^2 - 8x + 20$$

What are the zeros of function *f* defined by the above equation?

MATH REFERENCE FACT

The following statements have the same meaning as "c is a zero of function f":

- $x = c$ is a root or solution of the equation $f(x) = 0$.
- $x - c$ is a factor of $f(x)$.
- The graph of $y = f(x)$ intersects the x-axis at $(c, 0)$.

Solution

To find the zeros of function f, find the solutions of the equation $f(x) = 0$:

$$(2x^3 - 5x^2) + (-8x + 20) = 0$$
$$x^2(2x - 5) - 4(2x - 5) = 0$$
$$(2x - 5)(x^2 - 4) = 0$$
$$(2x - 5)(x - 2)(x + 2) = 0$$

Setting each factor equal to 0 yields $x = -2$, 2, and $\frac{5}{2}$.

The zeros of the function are **−2, 2, and** $\mathbf{\frac{5}{2}}$. The zeros of function f correspond to the x-intercepts of its graph in the xy-plane.

➥ Example

$$f(x) = 3x^3 + kx^2 - 32x + 28$$

The function f is defined by the equation above where k is a nonzero constant. In the xy-plane, the graph of f intersects the x-axis at three points: $(-2, 0)$, $\left(\frac{3}{2}, 0\right)$, and $(c, 0)$. What is the value of k?

(A) −25
(B) −17
(C) 7
(D) 14

Solution

- Since the x-intercepts correspond to the factors of a polynomial function, $f(x) = (x + 2)(3x - 2)(x - c)$. If the factors of function f are multiplied out, the constant term would be equal to the product of the constant terms in the three factors:

$$\begin{aligned}(2)(-2)(-c) &= 28\\ 4c &= 28\\ c &= 7\end{aligned}$$

- Multiply the factors of $f(x)$ together:

$$\begin{aligned}f(x) &= (x+2)(3x-2)(x-7)\\ &= (3x^2+4x-4)(x-7)\\ &= 3x^3-17x^2-32x+28\end{aligned}$$

- Since k is the coefficient of the x^2-term, $k = -17$.

The correct choice is **(B)**.

REMAINDERS AND FACTORS OF POLYNOMIALS

If $f(x) = x^2 - x - 6$, then $x - 3$ divides evenly into $f(x)$ since

$$\frac{x^2-x-6}{x-3} = \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} = x+2 \text{ with a remainder of } 0$$

Also, $f(3) = 3^2 - 3 - 6 = 0$. This illustrates that if $f(x)$ is divisible by $x - c$, then $f(c) = 0$.

REMAINDER AND FACTOR THEOREMS

If a polynomial $f(x)$ is divided by a binomial of the form $x - c$ where c is a constant, then the function value $f(c)$ is equal to the remainder.

- If $f(c) = 0$, then there is zero remainder so $f(x)$ is divisible by $x - c$ or, equivalently, $x - c$ is a factor of $f(x)$. This also means that the graph of $y = f(x)$ intersects the x-axis at $x = c$.
- If $f(c) \neq 0$, then $f(x)$ is not divisible by $x - c$ so it is not a factor of $f(x)$.

➡ Example

If $x + 3$ is a factor of $f(x) = px^2 + p^2x + 30$ and $p > 0$, what is the value of p?

Solution

Since $x + 3 = x - (-3)$ is a factor of the polynomial, $f(-3) = 0$:

$$
\begin{aligned}
f(-3) &= p(-3)^2 + p^2(-3) + 30 \\
0 &= 9p - 3p^2 + 30 \\
3p^2 - 9p - 30 &= 0 \\
p^2 - 3p - 10 &= 0 \\
(p - 5)(p + 2) &= 0 \\
p = 5 \quad &or \quad p = -2 \leftarrow \text{Reject since } p > 0
\end{aligned}
$$

The value of p is **5**.

SIMPLIFYING COMPLEX FRACTIONS

A **complex fraction** is a fraction in which its numerator, denominator, or both contain other fractions. To simplify a complex fraction, multiply its numerator and its denominator by the lowest common multiple (LCM) of all its denominators. To simplify

$$\frac{1+\frac{2}{x}}{1-\frac{4}{x^2}}$$

multiply the numerator and denominator by x^2:

$$\frac{1+\frac{2}{x}}{1-\frac{4}{x^2}} = \frac{x^2\left(1+\frac{2}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)}$$

$$= \frac{x^2+2x}{x^2-4}$$

$$= \frac{x\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$$

$$= \frac{x}{x-2}$$

➡ Example

When $x^{-1} - 1$ is divided by $x - 1$, the quotient is

(A) -1

(B) $-\frac{1}{x}$

(C) $\frac{1}{x^2}$

(D) $\frac{1}{(x-1)^2}$

Solution

$$\frac{x^{-1}-1}{x-1} = \frac{\frac{1}{x}-1}{x-1}$$

$$= \frac{x\left(\frac{1}{x}-1\right)}{x(x-1)}$$

$$= \frac{1-x}{x(x-1)}$$

$$= \frac{\overset{-1}{\cancel{1-x}}}{x\cancel{(x-1)}}$$

$$= -\frac{1}{x}$$

The correct choice is **(B)**.

SOLVING FRACTIONAL EQUATIONS

To solve an equation that contains algebraic fractions, clear the equation of its fractions by multiplying each term of the equation by the lowest common multiple of all of the denominators. To find the solution to the equation

$$\frac{4}{3}=\frac{-(3x+13)}{3x}+\frac{5}{6x}$$

first remove the parentheses by taking the opposite of each term inside the parentheses:

$$\frac{4}{3}=\frac{-3x-13}{3x}+\frac{5}{6x}$$

Eliminate the fractions by multiplying each term by $6x$, the lowest common multiple of all the denominators:

$$\begin{aligned}\overset{2x}{\cancel{6x}}\left(\frac{4}{\cancel{3}}\right)&=\overset{2}{\cancel{6x}}\left(\frac{-3x-13}{\cancel{3x}}\right)+\overset{1}{\cancel{6x}}\left(\frac{5}{\cancel{6x}}\right)\\ 2x(4)&=2(-3x-13)+5\\ 8x&=-6x-26+5\\ 8x+6x&=-21\\ \frac{14x}{14}&=\frac{-21}{14}\\ x&=-\frac{3}{2}\end{aligned}$$

➥ Example

$$\frac{2(n-1)}{3}-\frac{3(n+1)}{4}=\frac{n+3}{2}$$

In the equation above, what is the value of n^2?

Solution

Eliminate the fractional terms by multiplying each term of the equation by 12, the lowest common multiple of all the denominators:

$$\overset{4}{\cancel{12}}\left[\frac{2(n-1)}{\cancel{3}}\right]-\overset{3}{\cancel{12}}\left[\frac{3(n+1)}{\cancel{4}}\right]=\overset{6}{\cancel{12}}\left(\frac{n+3}{\cancel{2}}\right)$$

$$\begin{aligned}4\cdot 2(n-1)-3\cdot 3(n+1)&=6(n+3)\\ 8n-8-9n-9&=6n+18\\ -n-17&=6n+18\\ -n-6n&=18+17\\ \frac{-7n}{-7}&=\frac{35}{-7}\\ n&=-5\end{aligned}$$

Hence, $n^2 = (-5)^2 = \mathbf{25}$.

➡ Example

$$\frac{9}{y+1}+\frac{18}{y^2-1}=1$$

What is a possible solution to the equation above?

Solution

First factor the second denominator. Then eliminate the fractions by multiplying each term of the equation by the lowest common multiple of the denominators:

$$\frac{9}{y+1}+\frac{18}{(y+1)(y-1)}=1$$

$$[\cancel{(y+1)}(y-1)]\frac{9}{\cancel{y+1}}+[\cancel{(y+1)(y-1)}]\frac{18}{\cancel{(y+1)(y-1)}}=[(y+1)(y-1)]\cdot 1$$

$$\begin{aligned}9(y-1)+18&=y^2-1\\ 9y-9+18&=y^2-1\\ y^2-9y-10&=0\\ (y-10)(y+1)&=0\\ y-10=0 \quad &or \quad y+1=0\\ y=10 \ or \quad & y=-1 \longleftarrow \text{reject!}\end{aligned}$$

If $y = -1$, the denominator of the first fraction in the original equation evaluates to 0 so this solution is rejected. The only possible solution is $y =$ **10**.

LESSON 5-2 TUNE-UP EXERCISES

Multiple-Choice

1. The polynomial $x^3 - 2x^2 - 9x + 18$ is equivalent to

   (A) $(x - 9)(x - 2)^2$
   (B) $(x - 2)(x - 3)(x + 3)$
   (C) $(x + 3)(x - 2)^2$
   (D) $(x - 2)(x + 2)(x - 3)$

2. When resistors R^1 and R^2 are connected in a parallel electric circuit, the total resistance is

   $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$. This fraction is equivalent to

   (A) $R_1 + R_2$
   (B) $\frac{R_1 + R_2}{R_1 R_2}$
   (C) $\frac{R_1}{R_2} + \frac{R_2}{R_1}$
   (D) $\frac{R_1 R_2}{R_1 + R_2}$

3. In how many different points does the graph of the function $f(x) = x^3 - 2x^2 + x - 2$ intersect the x-axis?

   (A) 0
   (B) 1
   (C) 2
   (D) 3

$$\frac{x^2 + 9x - 22}{x^2 - 121} \div (2 - x)$$

4. The expression above is equivalent to

(A) $x - 11$

(B) $\frac{1}{x-11}$

(C) $11 - x$

(D) $\frac{1}{11-x}$

5. If $p(x)$ is a polynomial function and $p(4) = 0$, then which statement is true?

(A) $x + 4$ is a factor of $p(x)$.
(B) $x - 4$ is a factor of $p(x)$.
(C) The greatest power of x in $p(x)$ is 4.
(D) $p(x)$ is divisible by 4.

$$\left(\frac{9}{4}x^2 - 1\right) - \left(\frac{3}{2}x - 1\right)^2$$

6. The expression above is equivalent to

(A) $3x - 2$
(B) $-3x$
(C) $\frac{3}{4}x - 2$
(D) 0

$$\frac{\frac{x-y}{y}}{y^{-1} - x^{-1}}$$

7. The expression above is equivalent to

(A) x
(B) y
(C) $\frac{1}{y}$
(D) $-\frac{x}{y}$

$$f(x) = 3x^3 - 5x^2 - 48x + 80$$

8. If the zeros of function f defined above are represented by r, s, and t, what is the value of the sum $r + s + t$?

(A) $\frac{3}{5}$

(B) $\frac{5}{3}$

(C) $\frac{17}{3}$

(D) 8

$$\frac{y^3 + 3y^2 - y - 3}{y^2 + 4y + 3}$$

9. The expression above is equivalent to

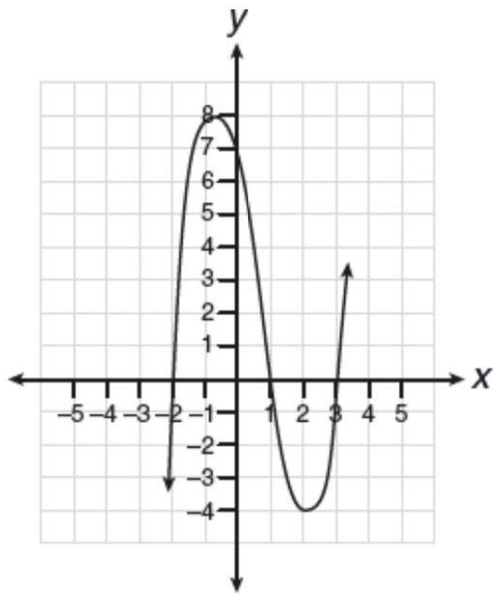
(A) $y - 1$
(B) $y + 1$
(C) $\frac{y-1}{y+3}$
(D) $y^2 - 1$

x	f(x)	g(x)
–3	3	0
–1	0	3
0	–4	4
2	0	–2

10. Several values of x, and the corresponding values for polynomial functions f and g are shown in the table above. Which of the following statements is true?

I. $f(0) + g(0) = 0$
II. $f(x)$ is divisible by $x + 2$
III. $g(x)$ is divisible by $x + 3$

(A) I, II, and III
(B) I and II, only
(C) II and III, only
(D) I and III, only



11. Which equation(s) represent(s) the graph above?

I. $y = (x + 2)(x^2 - 4x - 12)$
II. $y = (x - 3)(x^2 + x - 2)$
III. $y = (x - 1)(x^2 - 5x - 6)$

(A) I only
(B) II only
(C) I and II
(D) II and III

12. Which of the following functions have zeros −1, 1, and 4?

(A) $f(x) = (x - 4)(1 + x^2)$
(B) $f(x) = (x + 4)(1 - x^2)$
(C) $f(x) = (x - 1)(x^2 - 3x - 4)$
(D) $f(x) = (x - 1)(x^2 + 3x - 4)$

$$\left(\frac{10x^2y}{x^2 + xy}\right) \times \left(\frac{(x + y)^2}{2xy}\right) \div \left(\frac{x^2 - y^2}{y^2}\right)$$

13. Which of the following is equivalent to the expression above?

(A) $\dfrac{5y^2}{x-y}$

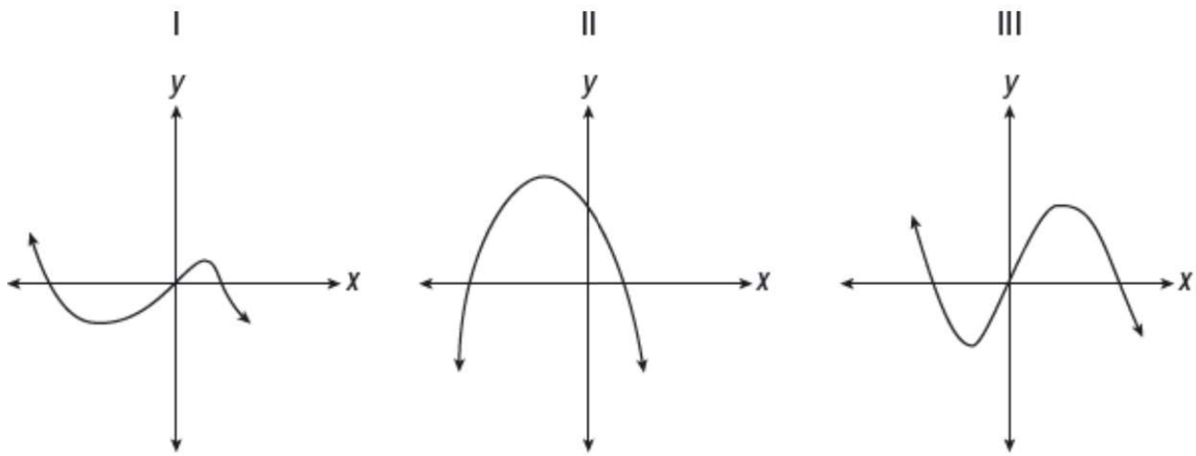
(B) $\dfrac{y^2}{x-y}$

(C) $\dfrac{xy}{x-y}$

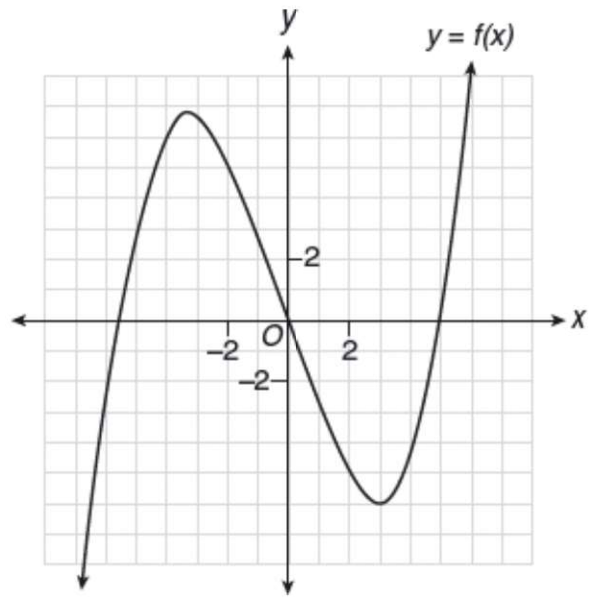
(D) $\dfrac{x+y}{xy}$

$$f(x) = (2 - 3x)(x + 3) + 4(x^2 - 6)$$

14. What is the sum of the zeros of function f defined by the equation above?
    (A) 3
    (B) 6
    (C) 7
    (D) 11



15. A polynomial function contains the factors x, $x - 2$, and $x + 5$. Which of the graph(s) above could represent the graph of this function?
    (A) I only
    (B) II only
    (C) III only
    (D) I and III



16–18 The graph of polynomial function *f* is shown above.

16. What is the greatest integer value of k for which $f(x) = k$ has exactly 3 real solutions?

(A) –5
(B) 0
(C) 6
(D) 7

17. What is the best estimate of the remainder when $f(x)$ is divided by $x + 3$?

(A) –6.0
(B) 0
(C) 6.5
(D) It cannot be determined.

18. What is the maximum number of points a circle whose center is at the origin can intersect the graph of $y = f(x)$?

(A) 2
(B) 3
(C) 4
(D) 6

$$(y^2 + ky - 3)(y - 4) = y^3 + by^2 + 5y + 12$$

19. In the equation above, k is a nonzero constant. If the equation is true for all values of y, what is the value of k?

(A) $\frac{1}{2}$
(B) -2
(C) 4
(D) 6

$$\frac{16a^4 - 81b^4}{8a^3 + 12a^2b + 18ab^2 + 27b^3}$$

20. Which of the following expressions is equivalent to the expression above?

(A) $4a^2b + 9ab^2 - a^2b^2$
(B) $4a^2b - 9ab^2$
(C) $2a + 3b$
(D) $2a - 3b$

Grid-In

$$\frac{k}{6} + \frac{3(1-k)}{4} = \frac{k-5}{2}$$

1. What is the solution for k in the equation above?

$$\frac{3}{2} = \frac{-(5m-3)}{3m} + \frac{7}{12m}$$

2. What is the solution for m in the equation above?

$$f(x) = x^3 + 5x^2 - 4x - 20$$

3. How many of the zeros of function f defined by the equation above are located in the interval $-4 \le x \le 4$?

$$\frac{t}{t-3} - \frac{t-2}{2} = \frac{5t-3}{4t-12}$$

4. If x and y are solutions of the equation above and $y > x$, what is the value of $y - x$?

$$x^3 + 150 = 6x^2 + 25x$$

5. What is the sum of all values of x that satisfy the equation above?

$$p(t) = t^5 - 3t^4 - kt + 7k^2$$

6. In the polynomial function above, k is a nonzero constant. If $p(t)$ is divisible by $t - 3$, what is the value of k?

LESSON 5-3 COMPLEX NUMBERS

OVERVIEW

The **imaginary unit** i is defined such that $i^2 = -1$ so that $i = \sqrt{-1}$. An **imaginary number** is the product of a nonzero real number and the imaginary unit i, as in $2i$.

COMPLEX NUMBERS

When an imaginary number is combined with a real number, as in $3 + 2i$, the result is called a *complex number*. A **complex number** is a number that has the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. Since any real number a can be written in the form $a + 0 \cdot i$, the set of complex numbers includes the set of real numbers. For example, *5* is a complex number since $5 = 5 + 0 \cdot i$.

Properties of Complex Numbers

All of the properties of arithmetic that work for real numbers also hold for complex numbers.

- When performing arithmetic operations with complex numbers, terms involving i are treated as if they are monomials:

  1. $3i + 5i = 8i$

  2. $6i - i = 5i$

  3. $i \cdot i^2 = i^{1+2} = i^3$

  4. $\frac{i^{13}}{i^4} = i^{13-4} = i^9$

- To add or subtract complex numbers of the form $a + bi$, combine the real parts and then combine the imaginary parts:

1. $(2 + 3i) + (4 - 5i) = (2 + 4) + (3i - 5i) = 6 - 2i$

2. $(1 - 2i) - (-4 + 6i) = (1 - 2i) + (4 - 6i)$
$= (1 + 4) + (-2i - 6i)$
$= 5 - 8i$

3. $\sqrt{-64} + 2\sqrt{-9} = \sqrt{64}\cdot\sqrt{-1} + 2\sqrt{9}\cdot\sqrt{-1}$
$= 8i + 6i$
$= 14i$

4. $\sqrt{-50} + 4\sqrt{-18} = \left(\sqrt{25}\cdot\sqrt{2}\cdot\sqrt{-1}\right) + \left(4\sqrt{9}\cdot\sqrt{2}\cdot\sqrt{-1}\right)$
$= 5\sqrt{2}i \qquad + 12\sqrt{2}i$
$= 17\sqrt{2}i$

- To multiply complex numbers of the form $a + bi$, treat the complex numbers as binomials and use the FOIL method.

$$\begin{aligned}
& \qquad\qquad\qquad\quad \text{F} \qquad\quad \text{O} \qquad\quad \text{I} \qquad\quad \text{L} \\
(3 + 2i)(5 - 4i) &= (5\cdot 3) + (3)(-4i) + (2i)(5) + (2i)(-4i) \\
&= 15 - 12i + 10i - 8i^2 \\
&= 15 - 2i - 8(-1) \\
&= 23 - 2i
\end{aligned}$$

Simplifying Higher Powers of *i*

The first few powers of *i* are worth remembering:

$$\begin{aligned}
i^0 &= 1 \\
i^1 &= i \\
i^2 &= -1 \\
i^3 &= -i
\end{aligned}$$

TIP

In simplifying higher powers of *i*, make use of these facts: $i^2 = -1$, -1 raised to an even power is 1, and -1 raised to an odd power is -1.

You can simplify a higher power of *i* by rewriting it so that it is expressed in terms of a power of i^2:

$$\begin{aligned} i^{35} &= i^{34} \cdot i && \leftarrow \text{Factor } i^{35} \text{ so that one of its factors has the greatest even exponent} \\ &= (i^2)^{17} \cdot i && \leftarrow \text{Rewrite } i^{34} \text{ as a power of } i^2 \\ &= (-1)^{17} \cdot i && \leftarrow \text{Let } i^2 = -1 \\ &= -i && \leftarrow \text{Simplify} \end{aligned}$$

To illustrate further,

$$\begin{aligned} i^{82} &= (i^2)^{41} \\ &= (-1)^{41} \\ &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} i^{13} &= i^{12} \cdot i^1 \\ &= (i^2)^6 \cdot i \\ &= (-1)^6 \cdot i \\ &= i \end{aligned}$$

MATH REFERENCE FACT

The product of two complex conjugates is always a positive real number:

$$(a + bi)(a - bi) = a^2 + b^2$$

COMPLEX CONJUGATES: $a \pm bi$

The numbers $3 + 2i$ and $3 - 2i$ are *complex conjugates*. A pair of complex numbers having the form $a + bi$ and $a - bi$ are **complex conjugates**. The sum and product of a pair of complex conjugates are always positive real numbers. For example,

$$\begin{aligned} (3 + 2i)(3 - 2i) &= 9 - 6i + 6i - 4i^2 \\ &= 9 - 4(-1) \\ &= 9 + 4 \\ &= 13 \end{aligned}$$

➥ Example

Which of the following complex numbers is equivalent to $\frac{8 - 4i}{5 + 3i}$? (Note: $i = \sqrt{-1}$)

(A) $\frac{14}{17}+\frac{22}{17}i$

(B) $\frac{14}{17}-\frac{22}{17}i$

(C) $\frac{13}{8}-\frac{24}{17}i$

(D) $\frac{13}{8}+\frac{24}{17}i$

Solution

In order to produce an equivalent fraction with a real number in the denominator, multiply the numerator and the denominator by $5 - 3i$, the complex conjugate of $5 + 3i$:

$$\begin{aligned}
\frac{8-4i}{5+3i}\cdot\frac{5-3i}{5-3i} &= \frac{(8-4i)(5-3i)}{25-9i^2} \\
&= \frac{40-24i-20i+12i^2}{25+9} \\
&= \frac{40-44i-12}{34} \\
&= \frac{28-44i}{34} \\
&= \frac{2(14-22i)}{34} \\
&= \frac{14-22i}{17} \\
&= \frac{14}{17}-\frac{22}{17}i
\end{aligned}$$

The correct choice is **(B)**.

LESSON 5-3 TUNE-UP EXERCISES

Multiple-Choice

NOTE: *Unless indicated otherwise,* $i = \sqrt{-1}$ *for each problem.*

1. Which of the following is equal to $i^{50} + i^0$?

   (A) 1
   (B) 2
   (C) –1
   (D) 0

2. Which of the following is equivalent to $2i^2 + 3i^3$?

   (A) $-2 - 3i$
   (B) $2 - 3i$
   (C) $-2 + 3i$
   (D) $2 + 3i$

3. Expressed in simplest form, $2\sqrt{-50} - 3\sqrt{-8}$ is equivalent to

   (A) $16i\sqrt{2}$
   (B) $3i\sqrt{2}$
   (C) $4i\sqrt{2}$
   (D) $-i\sqrt{2}$

4. If $x = 3i$, $y = 2i$, $z = m + i$, and $i = \sqrt{-1}$, then the expression $xy^2z =$

   (A) $-12 - 12mi$
   (B) $-6 - 6mi$
   (C) $12 - 12mi$
   (D) $6 - 6mi$

5. If $g(x) = \left(x\sqrt{1-x}\right)^2$, what is $g(10)$?

   (A) –30
   (B) –900
   (C) $30i$
   (D) $900i$

6. Which of the following is equal to $(x + i)^2 - (x - i)^2$?

(A) 0
(B) -2
(C) $-2 + 4xi$
(D) $4xi$

$$i^{13} + i^{18} + i^{31} + n = 0$$

7. In the equation above, what is the value of n in simplest form?

(A) $-i$
(B) -1
(C) 1
(D) i

8. Which of the following is equivalent to $2i(xi - 4i^2)$?

(A) $2x - 8i$
(B) $-2x + 8i$
(C) $-6xi$
(D) $-8xi$

9. If $x = 2i$, $y = -4$, $z = 3i$, and $i = \sqrt{-1}$, then $\sqrt{x^3yz} =$

(A) $4\sqrt{6}i$
(B) $24i$
(C) $-4\sqrt{6}$
(D) -24

10. Which of the following is equal to $(13 + 17i)(4 - 9i)$?

(A) -12
(B) 116
(C) $115 - 89i$
(D) $52 - 126i$

11. If $(x - yi) + (a + bi) = 2x$ and $i = \sqrt{-1}$, then $(x + yi)(a + bi) =$

(A) $x^2 + y^2$
(B) $x^2 - y^2$
(C) $4x^2 + y^2$

(D) $5x^2$

12. Which of the following complex numbers is equivalent to $\frac{3+i}{4-7i}$?

(A) $\frac{17}{28}$

(B) $-\frac{19}{33}-\frac{25}{33}i$

(C) $\frac{1}{13}-\frac{5}{13}i$

(D) $\frac{1}{13}+\frac{5}{13}i$

13. In an electrical circuit, the voltage, E, in volts, the current, I, in amps, and the opposition to the flow of current, called impedance, Z, in ohms, are related by the equation $E = IZ$. What is the impedance, in ohms, of an electrical circuit that has a current of $(3 + i)$ amps and a voltage of $(-7 + i)$ volts?

(A) $-2+i$

(B) $1-2i$

(C) $\frac{-11}{25}-\frac{1}{5}i$

(D) $-\frac{16}{25}i$

$$(9 + 2i)(4 - 3i) - (5 - i)(4 - 3i)$$

14. The expression above is equivalent to which of the following?

(A) 7
(B) $14 - 18i$
(C) 25
(D) $16 + 18i$

Grid-In

NOTE: *Unless indicated otherwise, $i = \sqrt{-1}$ for each problem.*

1. What is the value of $\left(\frac{1}{2}+i\sqrt{5}\right)\left(\frac{1}{2}-i\sqrt{5}\right)$?

$$\left(2-\sqrt{-25}\right)\left(-7+\sqrt{-4}\right)=x+yi$$

2. In the equation above, what is the value of y?

3. If $(1 - 3i)(7 + 5i + i^2) = a + bi$, what is the value of $a + b$?

4. If $\frac{6+4i}{1-3i}$, what is the value of $a + b$?

$$g(x)=a\sqrt{41-x^2}$$

5. Function g is defined by the equation above where a is a nonzero real constant. If $g(2i)=\sqrt{5}$, where $i=\sqrt{-1}$, what is the value of a?

LESSON 5-4 COMPLETING THE SQUARE

OVERVIEW

This section explains how to complete the square and shows how it can be used to solve quadratic equations including those that cannot be solved by factoring.

HOW TO COMPLETE THE SQUARE

Sometimes it is useful to complete the square within a quadratic expression so that it includes the square of a binomial.

- To complete the square of a quadratic trinomial of the form $x^2 + bx + c$, add and then subtract $\left(\frac{b}{2}\right)^2$. To complete the square within $x^2 + 10x + 7$, add and then subtract $\left(\frac{10}{2}\right)^2$:

$$x^2 + 10x + 7 = \underbrace{\left(x^2 + 10x + \boxed{25}\right)}_{\text{Perfect square}} + 7 - 25$$
$$= (x+5)^2 - 18$$

- To complete the square within a quadratic expression having the form $ax^2 + bx + c$, first factor out a from the variable terms:

$$\begin{aligned} 2x^2 - 12x + 5 &= 2\left(x^2 - 6x + \boxed{?}\right) + 5 \\ &= 2\left(x^2 - 6x + \boxed{9}\right) + 5 - 18 \\ &= 2(x-3)^2 - 13 \end{aligned}$$

Since 2×9 is being added to complete the square, 18 also needs to be subtracted in order to produce an equivalent expression.

SOLVING NONFACTORABLE QUADRATIC EQUATIONS

Any quadratic equation, including those that are not factorable, can be solved by putting them in the form $(x + p)^2 = k$ and then taking the square root of both sides of the equation. To solve $x^2 - 8x + 9 = 0$ by completing the square,

- Rewrite the equation as $x^2 - 8x = -9$ so that only terms involving x are on one side of the equation.
- Complete the square by adding $\left(-\frac{8}{2}\right)^2 = 16$ to both sides of the equation:

$$x^2 - 8x + 16 = -9 + 16$$
$$x^2 - 8x + 16 = 7$$
$$(x - 4)^2 = 7$$

- Take the square root of both sides of the equation:

$$\sqrt{(x-4)^2} = \pm\sqrt{7}$$
$$x - 4 = +\sqrt{7} \quad \text{or} \quad x - 4 = -\sqrt{7}$$
$$x = 4 + \sqrt{7} \qquad\qquad x = 4 - \sqrt{7}$$

SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

To solve a quadratic equation by completing the square,

- Rewrite the quadratic equation in the form $x^2 + bx = k$.
- Add the number that completes the square to both sides of the equation.
- Take the square root of both sides of the equation and solve for x.

The Quadratic Formula

If you need to find the solutions to a quadratic equation, you may find it easier to use the quadratic formula.

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0.$$

Since the quadratic formula is *not* included on the SAT formula reference sheet, you should memorize it. The quantity underneath the radical sign, $b^2 - 4ac$, is called the **discriminant**. The discriminant is the part of the quadratic formula that determines whether the solutions are real or not real, equal or unequal, rational or irrational.

- If $b^2 - 4ac < 0$, the solutions are imaginary (not real).
- If $b^2 - 4ac = 0$, the solutions are real, equal, and rational.
- If $b^2 - 4ac > 0$, the solutions are real and unequal. If $b^2 - 4ac$ is a perfect square, then the solutions are rational; otherwise, the solutions are irrational.

➡ Example

What are the solutions to $y^2 + 6y + 7 = 0$?

Solution

Since the quadratic trinomial $y^2 + 6y + 7$ is not factorable, solve by either completing the square or by using the quadratic formula:

METHOD 1: Rewrite the equation as $y^2 + 6y = -7$ and solve it by completing the square.

- Add $\left(\frac{6}{2}\right)^2 = 9$ to both sides of the equation:

$$y^2 + 6y + 9 = -7 + 9 \text{ so } (y + 3)^2 = 2$$

- Take the square root of both sides of the equation:

$$y + 3 = \pm\sqrt{2}$$

- Solve for y:

$$y = -3 + \sqrt{2} \quad \text{or} \quad y = -3 - \sqrt{2}$$

METHOD 2: Solve using the quadratic formula where $a = 1$, $b = 6$ and $c = 7$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 28}}{2} \\ &= \frac{-6 \pm \sqrt{8}}{2} \\ &= \frac{-6 \pm 2\sqrt{2}}{2} \\ &= \frac{2\left(-3 \pm \sqrt{2}\right)}{2} \\ &= -3 \pm \sqrt{2} \end{aligned}$$

➡ Example

Solve $4x^2 - 8x - 3 = 0$ by completing the square.

Solution

- Transpose the constant term, −3, to the right side of the equation and factor out 4 from the left side:

$$4(x^2 - 2x) = 3$$

- Complete the square by adding $\left(\frac{-2}{2}\right)^2 = 1$ inside the parentheses. Since this is being multiplied by 4, you must add 4 × 1 to the right side of the equation:

$$\begin{aligned} 4\left(x^2 - 2x + 1\right) &= 3 + 4 \\ (x - 1)^2 &= \frac{7}{4} \end{aligned}$$

- Solve for x by taking the square root of both sides of the equation:

$$\sqrt{(x-1)^2} = \pm\sqrt{\frac{7}{4}}$$

$$x-1 = \frac{\sqrt{7}}{2} \quad or \quad x-1 = -\frac{\sqrt{7}}{2}$$

$$x = 1+\frac{\sqrt{7}}{2} \qquad x = 1-\frac{\sqrt{7}}{2}$$

➡ Example

What is the smallest integral value of k for which the roots of $3x^2 + 8x - k = 0$ are real?

(A) –6
(B) –5
(C) 0
(D) 6

Solution

The roots of a quadratic equation are real when the discriminant is greater than or equal to 0. If $3x^2 + 8x - k = 0$, then $a = 3$, $b = 8$, and $c = -k$:

$$b^2 - 4ac \geq 0$$
$$8^2 - 4(3)(-k) \geq 0$$
$$64 + 12k \geq 0$$
$$12k \geq -64$$
$$k \geq -\frac{64}{12}$$
$$k \geq -5\frac{1}{3}$$

The smallest integral value of k that satisfies the inequality is −5.

The correct choice is **(B)**.

SUM AND PRODUCT OF ROOTS

A quick way of finding the sum or product of the solutions to a quadratic equation without actually solving the equation is to use these formulas:

SUM AND PRODUCT OF ROOTS FORMULAS

If $ax2 + bx + c = 0$, then

Sum of two roots: $-\frac{b}{a}$ and Product of two roots: $\frac{c}{a}$

➡ Example

By what amount does the product of the solutions of $3x^2 - 10x + 13 = 0$ exceed the sum of its solutions?

Solution

- Sum of the solutions:

$$-\frac{b}{a} = -\left(\frac{-10}{3}\right) = \frac{10}{3}$$

- Product of the solutions:

$$\frac{c}{a} = \frac{13}{3}$$

- Product of the solutions exceeds their sum by $\frac{13}{3} - \frac{10}{3} = \frac{3}{3} = 1$.

LESSON 5-4 TUNE-UP EXERCISES

Multiple-Choice

1. What are the solutions to $3x^2 - 33 = 18x$?

(A) $x = 3 \pm 2\sqrt{5}$

(B) $x = \frac{3 \pm \sqrt{5}}{2}$

(C) $x = 3 \pm 4\sqrt{5}$

(D) $x = 3 \pm \frac{\sqrt{5}}{2}$

2. If the solutions to $2x^2 - 8x - 5 = 0$ are p and q with $p > q$, what is the value of $p - q$?

(A) $\sqrt{26}$

(B) $\frac{7}{2}$

(C) $2\sqrt{13}$

(D) $\frac{11}{2}$

$$\frac{x+5}{4} = \frac{1-x}{3x-4}$$

3. If the solutions to the equation above are r and s with $r > s$, what is the value of $r - s$?

(A) $\sqrt{7}$

(B) $\frac{5}{2}$

(C) $\sqrt{57}$

(D) 5

4. If the equation $y = 3x^2 + 18x - 13$ is written in the form $y = a(x - h)^2 + k$, what are the values of h and k?

(A) $h = 3, k = 14$
(B) $h = -3, k = -40$
(C) $h = 3, k = -13$

(D) $h = -3, k = -22$

$$x^2 + 6x + y^2 - 8y = 56$$

5. If the above equation is written in the form $(x - h)^2 + (y - k)^2 = r^2$, what is the value of r?

(A) 6
(B) 8
(C) 9
(D) $\sqrt{31}$

$$\frac{4}{x-3} + \frac{2}{x-2} = 2$$

6. If the solutions of the equation above in simplest radical form are $x = a \pm \sqrt{b}$, what are the values of a and b?

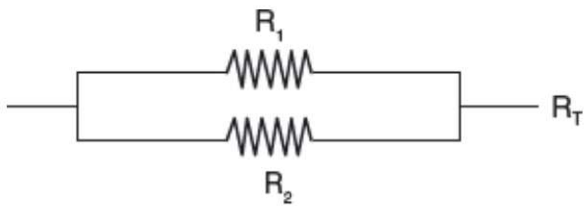
(A) $a = 4, b = 3$
(B) $a = -4, b = 5$
(C) $a = 3, b = 5$
(D) $a = -3, b = 5$

7. Which quadratic equation has $2 + 3i$ and $2 - 3i$ as its solutions?

(A) $x^2 + 4x - 13 = 0$
(B) $x^2 - 4x + 13 = 0$
(C) $x^2 + 13x - 4 = 0$
(D) $x^2 - 13x - 4 = 0$

8. The equation $ax^2 + 6x - 9 = 0$ will have imaginary roots if

(A) $a < -1$
(B) $a \geq -1$
(C) $a \leq 1$
(D) $-1 < a < 1$



$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_T}$$

9. If electrical circuits are hooked up in parallel, the reciprocal of the total resistance in the series is found by adding the reciprocals of each resistance as shown in the diagram above. In a certain circuit, R^2 exceeds the resistance of R^1 by 2 ohms, and the total resistance, R^T, is 1.5 ohms. Which expression represents the number of ohms in R^1?

(A) $\sqrt{13} - 1$

(B) $\sqrt{11} - 1$

(C) $\dfrac{1 + \sqrt{11}}{2}$

(D) $\dfrac{1 + \sqrt{13}}{2}$

10. The amount of water remaining in a certain bathtub as it drains when the plug is pulled is represented by the equation $L = -4t^2 - 8t + 128$, where L represents the number of liters of water in the bathtub and t represents the amount of time, in minutes, since the plug was pulled. Which expression represents the number of minutes it takes for half of the water that was in the bathtub before the plug was pulled to drain?

(A) $-1 + \sqrt{33}$

(B) $-1 + \sqrt{17}$

(C) $\dfrac{-1 + \sqrt{33}}{2}$

(D) $\dfrac{-1 + 2\sqrt{17}}{2}$

LESSON 5-5 THE PARABOLA AND ITS EQUATIONS

OVERVIEW

The graph of the quadratic function $y = ax^2 + bx + c$ is a **U**–shaped curve called a **parabola** with c as its y-intercept.

- If the constant $a > 0$, the parabola opens up  and the turning point or vertex of the parabola is the lowest point on the curve.
- If the constant $a < 0$, the parabola opens down  and the vertex of the parabola is the highest point on the curve.

VERTEX OF A PARABOLA

A line can be drawn through a parabola that divides it into two mirror image parts. The line of symmetry intersects the parabola at its **vertex**. The vertex is either the highest or the lowest point on the parabola. For the parabola $y = ax^2 + bx + c$,

- If $a > 0$, the parabola opens up and the vertex is the lowest (minimum) point on the curve. See Figure 5.1. The graph of $y = x^2 - x - 6$ is a parabola that opens up since the coefficient of the x^2-term is positive.

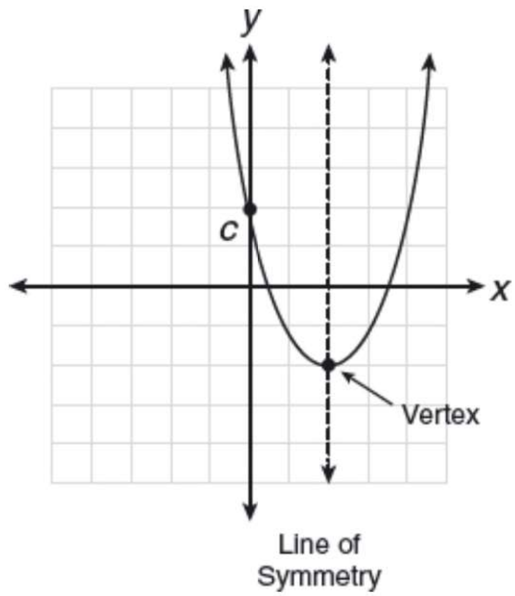


Figure 5.1 Graph of $y = ax^2 + bx + c$ with $a > 0$

- If $a < 0$, the parabola opens down and the vertex is the highest (maximum) point on the curve. See Figure 5.2. The graph of $y = -2x^2 - x + 8$ is a parabola that opens down since the coefficient of the x^2-term is negative.

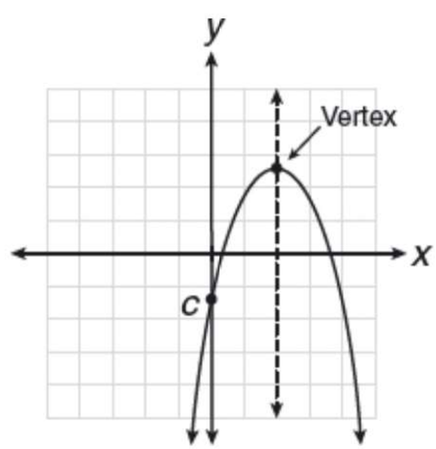


Figure 5.2 Graph of $y = ax^2 + bx + c$ with $a < 0$

EQUATION OF LINE OF SYMMETRY

If a parabola equation is given in the standard $y = ax^2 + bx + c$ form, you can use the formula $x = -\frac{b}{2a}$ to find an equation of the line of symmetry, which also gives the x-coordinate of the vertex. If $y = 2x^2 - 12x + 3$, then an equation of the line of symmetry is

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-12}{2(2)} \\ &= 3 \end{aligned}$$

Since the x-coordinate of the vertex is 3, you can find the y-coordinate of the vertex by substituting 3 for x in the parabola equation:

$$\begin{aligned} y &= 2x^2 - 12x + 3 \\ &= 2(3)^2 - 12(3) + 3 \\ &= 18 - 36 + 3 \\ &= -15 \end{aligned}$$

The vertex is at (3, −15). Since the vertex is a minimum point on the parabola, −15 represents the minimum value of the function $f(x) = 2x^2 - 12x + 3$.

MATCHING UP PAIRS OF PARABOLA POINTS

Each point on the parabola, other than the vertex, has a matching point on the other side of the line of symmetry that is the same distance from it. Figure 5.3 shows the graph of $f(x) = x^2 - 10x + 27$ with $x = 5$ as its line of symmetry. The point (3, 6) is on the parabola and 2 units from the line of symmetry. The matching point on the parabola is (7, 6), which lies on the opposite side of the line of symmetry and is also 2 units from it. Using function notation, $f(3) = f(7)$. Similarly, $f(4) = f(6)$.

TIP

A parabola's line of symmetry is the perpendicular bisector of every horizontal segment that connects two points on the parabola.

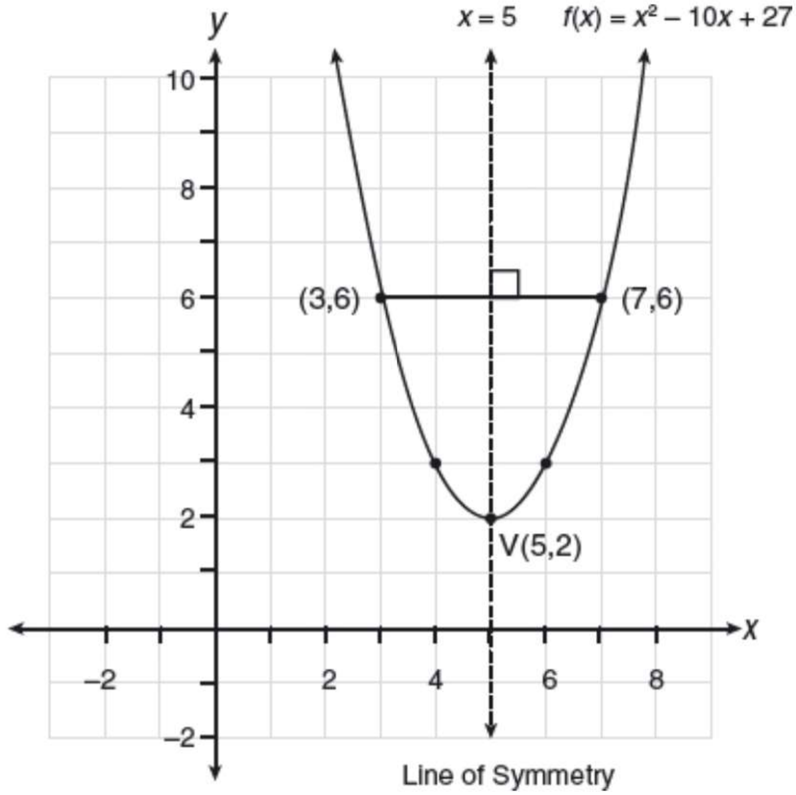


Figure 5.3 Matching up pairs of points on a parabola

VERTEX FORM OF PARABOLA EQUATION: $y = a(x - h)^2 + k$

The parabola equation $y = ax^2 + bx + c$ can also be written in the **vertex form** $y = a(x - h)^2 + k$ where (h, k) are the coordinates of the vertex. The constant a has the same meaning in both forms of the parabola equation. If $y = 3(x - 4)^2 + 7$, then you know the parabola opens up since $3 > 0$; the vertex is at (4, 7) and is a minimum point on the graph. An equation of the line of symmetry is $x = 4$.

REWRITING PARABOLA EQUATIONS

The SAT may ask you to rewrite the equation of a parabola in order to reveal certain information about it.

- To write an equation of a parabola so that its x-intercepts appear as constants in the equation, write the parabola equation in factored form. By rewriting the parabola equation $y = x^2 - 2x + 8$ as $y = (x - 2)(x - 4)$, its x-intercepts appear as constants.
- To write an equation of a parabola so that its minimum or maximum value appears as a constant, write the parabola equation in vertex form. When the parabola equation $y = x^2 - 2x + 8$ is written in the equivalent form $y = (x - 1)^2 + 7$, the minimum value of the function, 7, appears as a constant.

➡ Example

The graph of the function $f(x) = -\frac{1}{2}(x+4)(x+8)$ in the xy-plane is a parabola. Which of the following is an equivalent form of function f in which the maximum value of the function appears as a constant?

(A) $f(x) = (x+4)\left(4 - \frac{1}{2}x\right)$

(B) $f(x) = -\frac{1}{2}(x+6)^2 + 2$

(C) $f(x) = -\frac{1}{2}x(x+12) + 16$

(D) $f(x) = \frac{1}{2}(x+6)^2 + 20$

Solution

The maximum (or minimum) value of a quadratic function is the y-coordinate of the vertex of its graph which appears as the constant k in the vertex form of the equation $f(x) = a(x - h)^2 + k$.

- Using FOIL, multiply the binomial factors of $f(x)$, which gives

$$f(x) = -\frac{1}{2}(x^2 + 12x + 32)$$
$$= -\frac{1}{2}(x^2 + 12x) - \frac{1}{2}(32)$$
$$= -\frac{1}{2}(x^2 + 12x) - 16$$

- Change to the vertex form of the parabola equation by completing the square:

$$f(x) = -\frac{1}{2}(x^2 + 12x + 36) - 16 + \boxed{18} \leftarrow \text{Add 18 to balance adding} \left(-\frac{1}{2} \times 36\right)$$

$$= -\frac{1}{2}(x + 6)^2 + 2$$

- By comparing $f(x) = -\frac{1}{2}(x + 6)^2 + 2$ to $f(x) = a(x - h)^2 + k$, you know that $a = -\frac{1}{2}$, $h = -6$, and $k = 2$. Since $a < 0$, the vertex of the parabola is a maximum point so that the maximum value of f is 2.

The correct choice is **(B)**.

➡ Example

$$y = -x^2 + 120x - 2{,}000$$

The equation above gives the profit in dollars, y, a coat manufacturer earns each day where x is the number of coats sold. What is the maximum profit he earns in dollars?

Solution

- Because the leading coefficient of the parabola equation is negative, the vertex of the parabola is the highest point on the graph.
- First, determine the value of x that maximizes the profit by determining the coordinate of the vertex:

$$x = -\frac{b}{2a}$$

$$= -\frac{120}{2(-1)}$$

$$= 60$$

- The coat manufacturer must sell 60 coats each day to earn a maximum profit for that day. Because the maximum profit corresponds to the y-

coordinate of the vertex, substitute 60 for x in the parabola equation to find the corresponding value of y:

$$\begin{aligned} y &= -60^2 + 120(60) - 2{,}000 \\ &= -3{,}600 + 7{,}200 - 2{,}000 \\ &= 1{,}600 \end{aligned}$$

The maximum profit is **\$1,600**. You could also have arrived at the correct answer by changing the parabola equation to vertex form.

➡ Example

Stacy has 30 meters of fencing that she wishes to use to enclose a rectangular garden. If all of the fencing is used, what is the maximum area of the garden, in square meters, that can be enclosed?

(A) 48.75
(B) 56.25
(C) 60.50
(D) 168.75

Solution

- If x represents the length of the enclosed rectangular garden and w represents its width, then all 30 meters of fencing are used when $2x + 2w = 30$. Simplifying makes $x + w = 15$ so $w = 15 - x$. If $A(x)$ represents the area of the enclosed rectangle as a function of x, then

$$A(x) = xw = x(15 - x) = -x^2 + 15x$$

- Find the x-coordinate of the vertex:

$$x = -\frac{b}{2a} = -\frac{15}{2(-1)} = 7.5$$

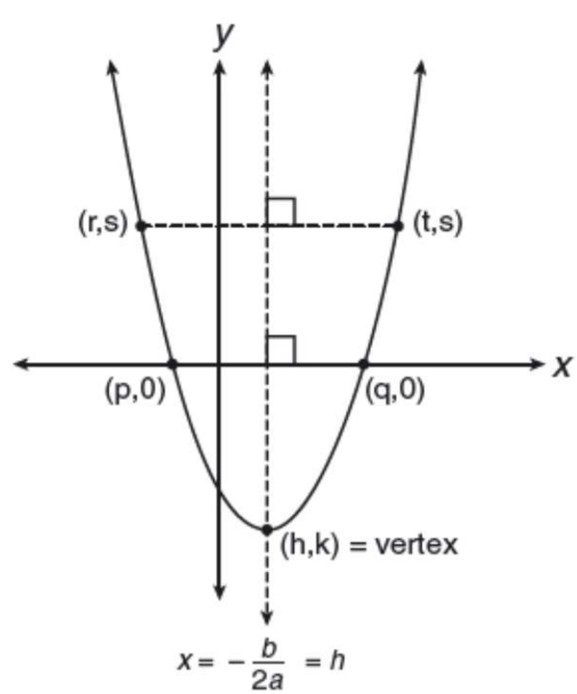
- The maximum value of $A(x)$ occurs at $x = 7.5$:

$$\begin{aligned} A(7.5) &= -(7.5)^2 + (15)(7.5) \\ &= -56.25 + 112.5 \\ &= 56.25 \text{ m}^2 \end{aligned}$$

The correct choice is **(B)**. You could also have arrived at the correct answer by changing the parabola equation to vertex form.

Table 5.2 summarizes some parabola facts you need to remember. In each parabola equation, the sign of the x^2-term determines whether the vertex at (h, k) is a maximum point (if $a < 0$) or a minimum point (if $a > 0$).

Table 5.2 Some Parabola Facts

Parabola Facts	Parabola ($a > 0$) $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$
• If $y = ax^2 + bx + c$, an equation of the line of symmetry is $x = -\frac{b}{2a}$ • If $f(x) = a(x - h)^2 + k$, the vertex is (h, k), and an equation of the line of symmetry is $x = h$. If $a > 0$, k is the minimum value of the function; if $a < 0$, k is the maximum value of the function. • The line of symmetry is the perpendicular bisector of any horizontal segment whose endpoints are on the parabola: $h = \frac{p+q}{2}$ and $h = \frac{r+t}{2}$	

MODELING PROJECTILE MOTION

Projectile motion that is under the influence of gravity, as when a ball is tossed in the air, has a parabola-shaped flight path in which the x-coordinate of each point on the curve represents how much time has elapsed after the object was launched and the corresponding y-coordinate gives the height of the object at that instant of time. The vertex of the parabola corresponds to the point at which the object reaches its maximum height.

➥ Example

$$h(t) = 144t - 16t^2$$

The function above represents the height, in feet, a ball reaches t seconds after it is tossed in the air from ground level.

a. What is the maximum height of the ball?

b. After how many seconds will the ball hit the ground before rebounding?

Solution

a. The maximum height of the ball corresponds to the y-coordinate of the vertex.

- The x-coordinate of the vertex is

$$-\frac{b}{2a} = \frac{-144}{2(-16)} = 4.5$$

- The y-coordinate of the vertex is

$$\begin{aligned} h(4.5) &= 144(4.5) - 16(4.5)^2 \\ &= 324 \end{aligned}$$

The maximum height is **324** feet.

b. When the ball hits the ground, $h(t) = 0$:

$$\begin{aligned} 144t - 16t^2 &= 0 \\ 4t(36 - 4t) &= 0 \\ 4t = 0 \quad &or \quad 36 - 4t = 0 \\ t = \frac{0}{4} \quad & \quad -4t = -36 \\ t = 0 \quad & \quad t = \frac{-36}{-4} = 9 \text{ seconds} \end{aligned}$$

The solution $t = 0$ seconds represents the instant of time at which the ball is tossed in the air. The solution $\boldsymbol{t = 9}$ seconds is the number of seconds it takes for the ball to reach the ground after it is launched.

SOLVING NONLINEAR SYSTEMS OF EQUATIONS

A linear quadratic system can be solved algebraically by solving the first degree equation for either variable and then substituting for that variable in the second degree equation. To solve the system $y = -x^2 + 4x - 3$ and $x + y = 1$ algebraically,

- Solve the linear equation for y that gives $y = 1 - x$. Eliminate y in the quadratic equation by replacing it with $1 - x$. The result is $1 - x = -x^2 + 4x - 3$, which simplifies to $x^2 - 5x + 4 = 0$.
- Solve $x^2 - 5x + 4 = 0$ by factoring:

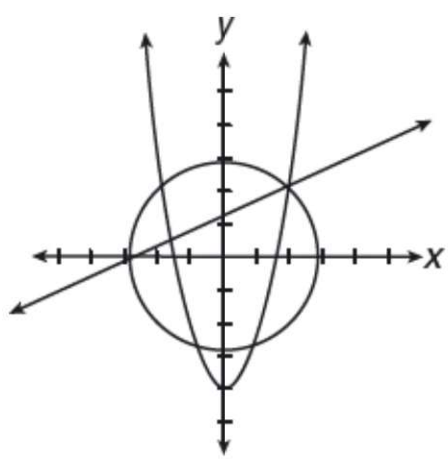
$$\begin{aligned} (x-1)(x-4) &= 0 \\ x - 1 = 0 \quad &or \quad x - 4 = 0 \\ x = 1 \quad & \quad\quad x = 4 \end{aligned}$$

Find the corresponding values of y by substituting each solution for x in the linear equation:

- If $x = 1$, then $1 + y = 1$ and $y = 0$ so $(1, 0)$ is a solution.
- If $x = 4$, then $4 + y = 1$ and $y = -3$ so $(4, -3)$ is a solution.

When a nonlinear system of equations is graphed in the xy-plane, the points of intersection common to *all* of the equations, if any, represent the solutions to the system of equations.

➡ Example



A system of three equations whose graphs in the xy-plane are a line, a circle, and a parabola are shown above. How many solutions does the system have?

(A) 0
(B) 1
(C) 2
(D) 4

Solution

The solution to the system of equations is the common point or points at which the line, circle, and parabola all intersect. There is exactly one such point which is in the first quadrant.

The correct choice is **(B)**.

LESSON 5-5 TUNE-UP EXERCISES

Multiple-Choice

1. An archer shoots an arrow into the air such that its height at any time, t, is given by the function $h(t) = -16t^2 + kt + 3$. If the maximum height of the arrow occurs at 4 seconds after it is launched, what is the value of k?

   (A) 128
   (B) 64
   (C) 8
   (D) 4

2. A model rocket is launched vertically into the air such that its height at any time, t, is given by the function $h(t) = -16t^2 + 80t + 10$. What is the maximum height attained by the model rocket?

   (A) 140
   (B) 110
   (C) 85
   (D) 10

3. When a ball is thrown straight up at an initial velocity of 54 feet per second. The height of the ball t seconds after it is thrown is given by the function $h(t) = 54t - 12t^2$. How many seconds after the ball is thrown will it return to the ground?

   (A) 9.2
   (B) 6
   (C) 4.5
   (D) 4

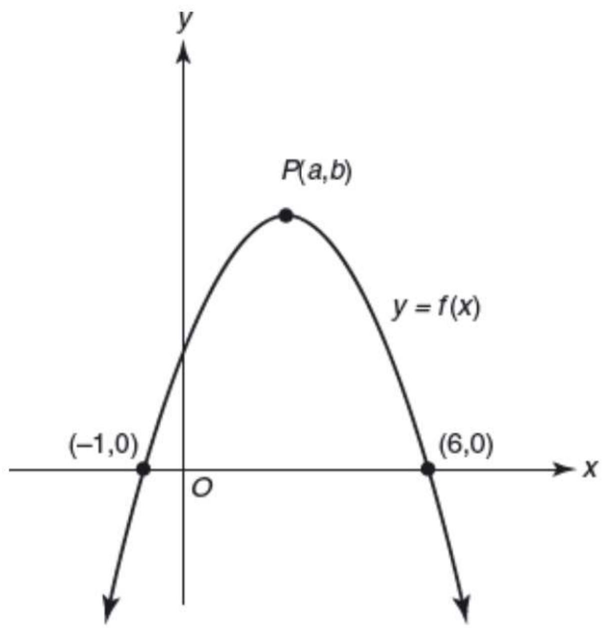
4. The graph of $y + 3 = (x - 4)^2 - 6$ is a parabola in the xy-plane. What are the x-intercepts of the parabola?

   (A) 1 and 7
   (B) -1 and -7
   (C) 4 and -6
   (D) 4 and -9

5. The graph of $y = (2x - 4)(x - 8)$ in the xy-plane is a parabola. Which of the following are true?

I. The graph's line of symmetry is $x = 5$
II. The minimum value of y is -7
III. The y-intercept of the graph is 32

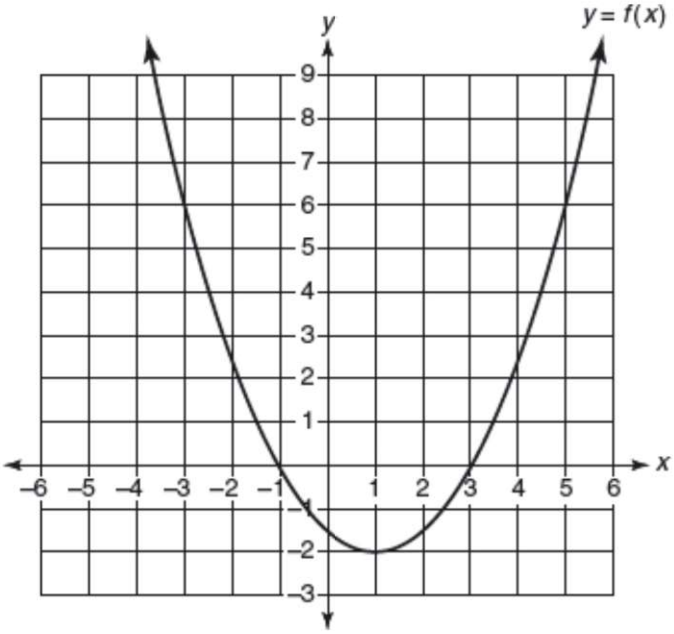
(A) I and II only
(B) I and III only
(C) II and III only
(D) I, II, and III



Note: Figure not drawn to scale

6. The graph of a quadratic function f is shown in the above figure. If $f(x) \leq b$ for all values of x, which of the following could be the coordinates of point P?

(A) (1.5, 2)
(B) (2.25, 3.5)
(C) (2.5, 4)
(D) (2.75, 5)



7. The figure above shows the graph of a quadratic function f with a minimum point at (1, −2). If $f(5) = f(c)$, then which of the following could be the value of c?

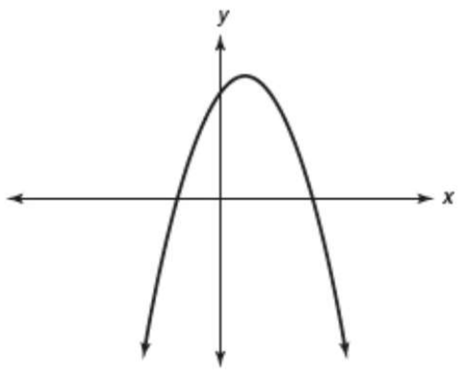
(A) −5
(B) −3
(C) 0
(D) 6

8. The graph of a quadratic function f intersects the x-axis at $x = -2$ and $x = 6$. If $f(8) = f(p)$, which could be the value of p?

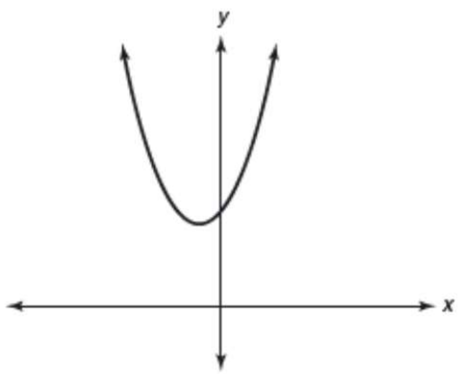
(A) −6
(B) −4
(C) −2
(D) 0

9. If in the quadratic function $f(x) = ax^2 + bx + c$, a and c are both negative constants, which of the following could be the graph of function f?

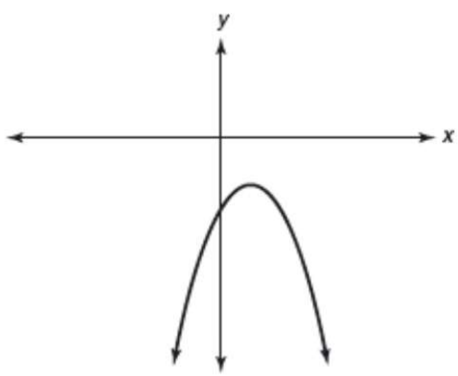
(A)



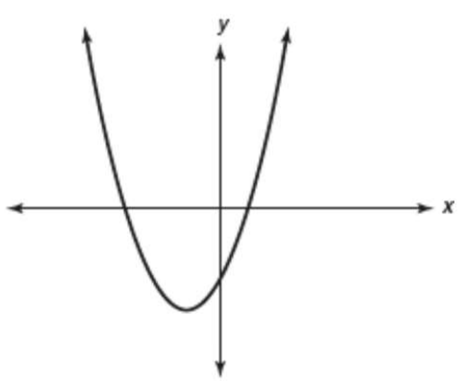
(B)



(C)



(D)



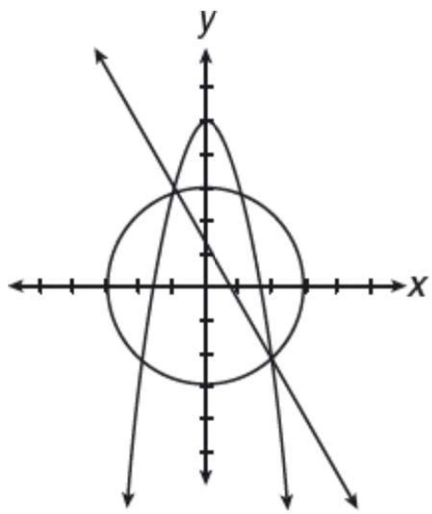
10. A parabola passes through the points (0, 0) and (6, 0). If the turning point of the parabola is $T(h, 4)$, which statement must be true?

I. $h = 2$

II. If the parabola passes through (1, 2), then it must also pass through (5, 2)

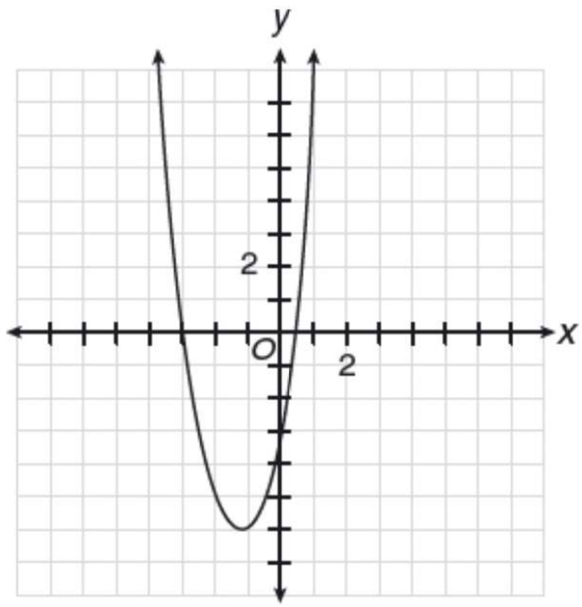
III. Point T is the highest point of the parabola

(A) II only
(B) III only
(C) I and II only
(D) II and III only



11. A system of three equations whose graphs in the xy-plane are a line, a circle, and a parabola is shown above. How many solutions does the system have?

(A) 1
(B) 2
(C) 3
(D) 4



12. Which of the following could be the equation of the graph above?

(A) $y = (x - 3)(2x + 1)$
(B) $y = (x + 3)(2x - 1)$
(C) $y = -(x - 3)(1 + 2x)$
(D) $y = \frac{1}{2}(x + 3)(x - 1)$

$$y = 2x^2 - 12x + 11$$

13. The graph of the equation above is a parabola in the xy-plane. What is the distance between the vertex of the parabola and the point (3, 1)?

(A) 1
(B) 8
(C) 10
(D) 12

$$f(x) = ax^2 + bx + c,\ a > 0$$

14. The coordinates of the lowest point on the graph of the function defined by the equation above is (3, 2). If $f(-1) = p$, then which of the following represents the value of p?

(A) $f(-5)$
(B) $f(-4)$

(C) $f(6)$
(D) $f(7)$

15. The parabola whose equation is $y = ax^2 + bx + c$ passes through the points $(-3, -40)$, $(0, 29)$, and $(-1, 10)$. What is an equation of the line of symmetry?

(A) $x = \frac{17}{4}$
(B) $x = \frac{9}{2}$
(C) $x = 5$
(D) $x = 6$

$$x^2 + y^2 = 416$$
$$y + 5x = 0$$

16. If (x, y) is a solution to the system of equations above and $x > 0$, what is the value of the difference $x - y$?

(A) 4
(B) 16
(C) 20
(D) 24

$$h(t) = -4.9t^2 + 68.6t$$

17. The function above gives the height of a model rocket, in meters, t seconds after it is launched from ground level. What is the maximum height, to the *nearest meter*, attained by the model rocket?

(A) 90
(B) 120
(C) 180
(D) 240

$$y = k(x - 1)(x + 9)$$

18. The graph of the equation above is a parabola in the xy-plane. If $k > 0$, what is the minimum value of y expressed in terms of k?

(A) $-7k$
(B) $-16k$
(C) $-25k$
(D) $-73k$

Grid-In

$$x^2 - y^2 = 18$$
$$y = x - 4$$

1. In the above system of equations, what is the value of $x + y$?

$$d(t) = -16t^2 + 40t + 24$$

2. A swimmer dives from a diving board that is 24 feet above the water. The distance, in feet, that the diver travels after t seconds have elapsed is given by the function above. What is the maximum height above the water, in feet, the swimmer reaches during the dive?

3. The marketing department at Sports Stuff found that approximately 600 pairs of running shoes will be sold monthly when the average price of each pair of running shoes is \$90. It was observed that for each \$5 reduction in price, an additional 50 pairs of running shoes will be sold monthly. What price per pair of running shoes will maximize the store's monthly revenue from the sale of running shoes?

Questions 4–6 refer to the equation below.

$$h(x) = -\frac{1}{225}x^2 + \frac{2}{3}x$$

The function h above models the path of a football when it is kicked during an attempt to make a field goal where x is the horizontal distance, in feet, from the kick, and $h(x)$ is the number of feet in the corresponding height of the football above the ground.

4. After the ball is kicked, what is the number of feet the football travels horizontally before it hits the ground?

5. What is the number of feet in the maximum height of the football?

6. The goal post is 10 feet high and a horizontal distance of 45 yards from the point at which the ball is kicked. By how many feet will the football fail to pass over the goal post?

LESSON 5-6 REFLECTING AND TRANSLATING FUNCTION GRAPHS

OVERVIEW

Changing the equation of a function in a certain way can have predictable effects on moving its graph in the xy-plane without changing its size or shape. A **reflection** flips a graph over a line so that the original and reflected graphs are "mirror images." A **translation** moves a graph sideways or up/down, or both sideways and up/down.

REFLECTING GRAPHS OF FUNCTIONS

Inserting a negative sign within the parentheses of $y = f(x)$ flips its graph over the y-axis, whereas writing a negative sign outside the parentheses in front of the function flips its graph upside down over the x-axis. Reflecting a graph does not change its size or shape. See Figure 5.4.

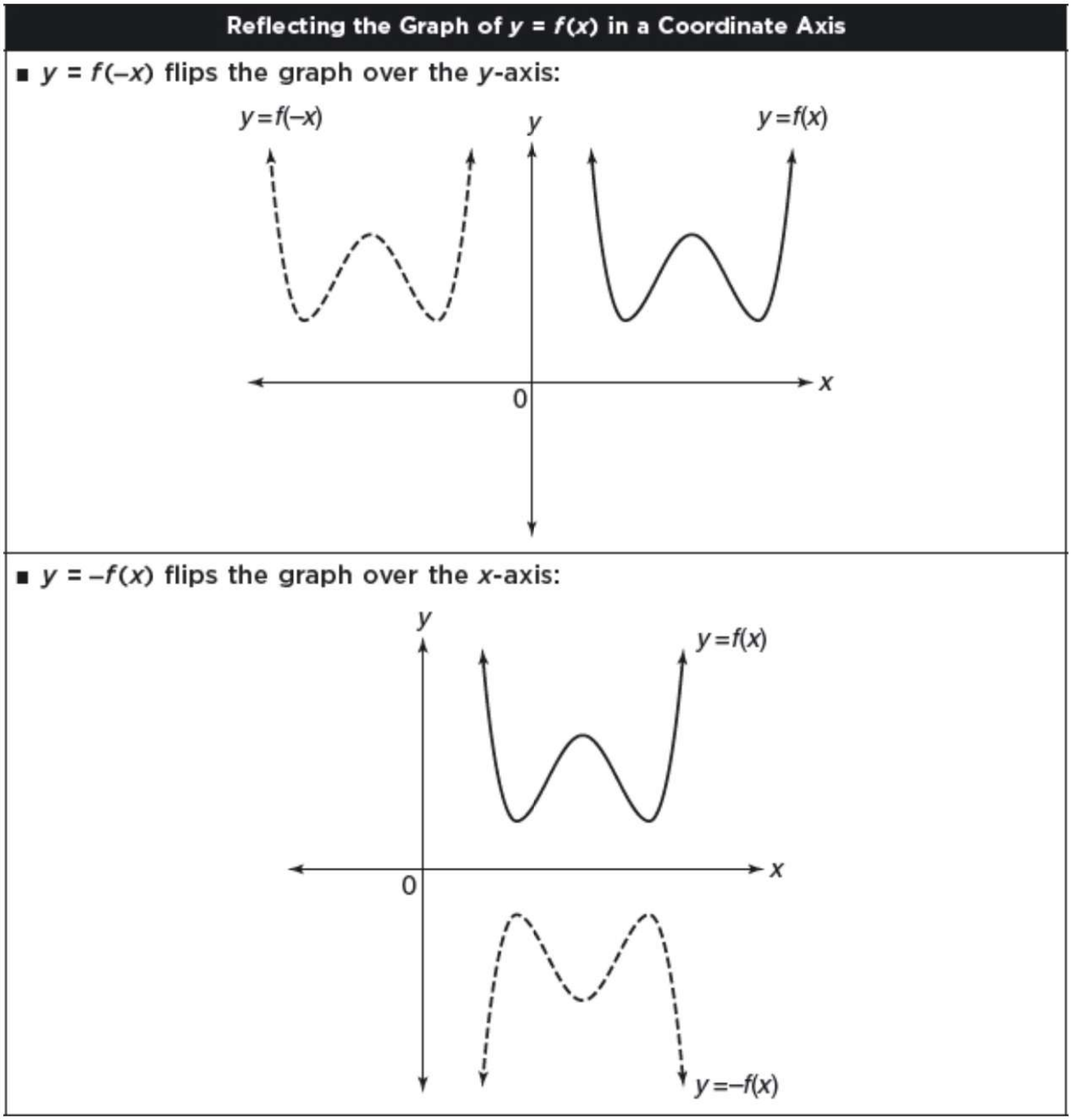


Figure 5.4 Reflecting a function graph in a coordinate axis in the *xy*-plane

TRANSLATING GRAPHS OF FUNCTIONS

Translating a graph in the *xy*-plane moves the graph sideways or up/down without rotating it, or changing its size or shape. See Figure 5.5.

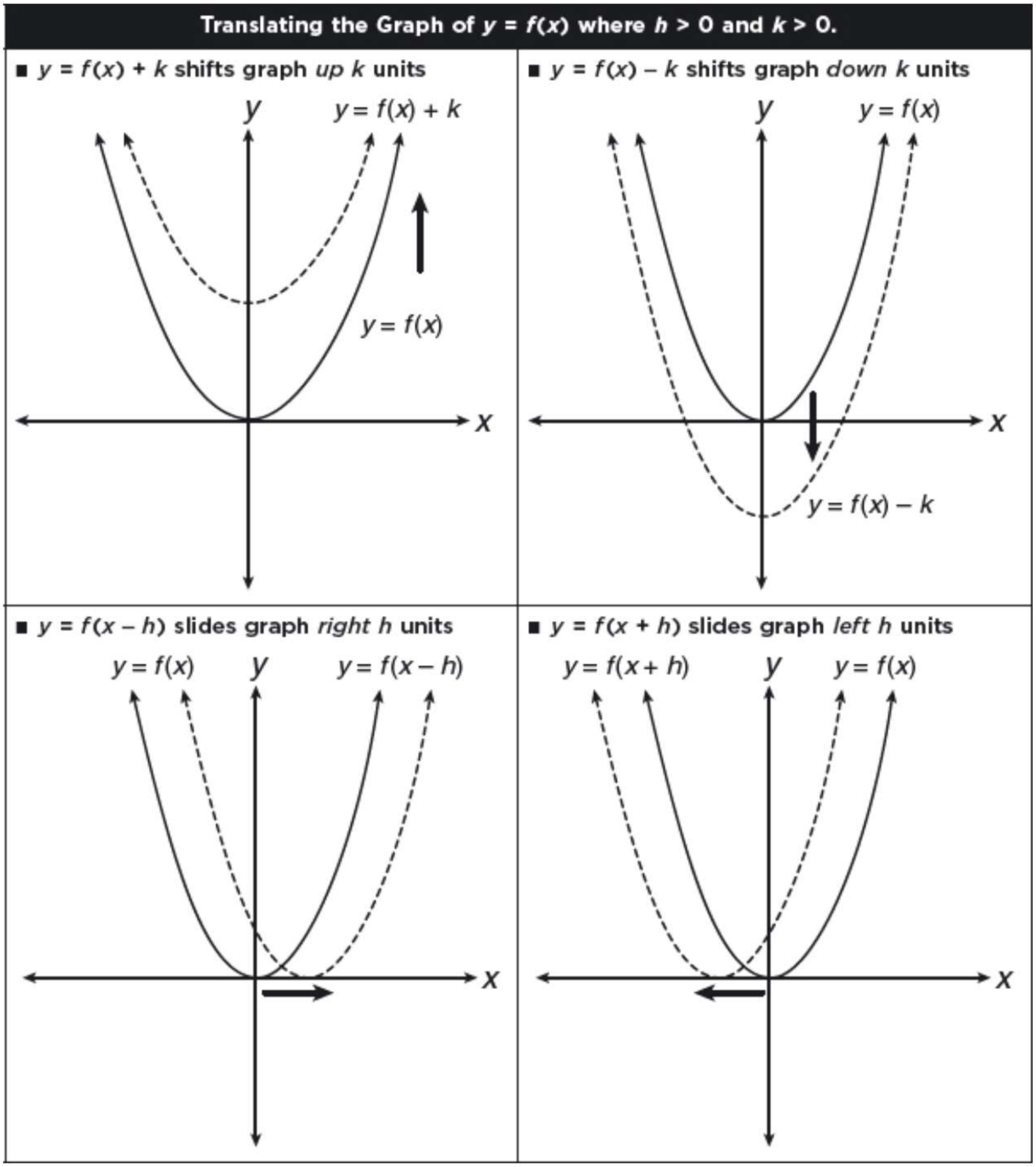


Figure 5.5 Translating a function graph in the *xy*-plane

TIP

- $y = f(x) + k$: Adding k outside the parentheses shifts the graph of function f vertically k units.

- $y = f(x + h)$: Adding h *inside* the parentheses shifts the graph of function f sideways h units.

Here are some examples:

- $y = -\sqrt{x}$ flips the graph of $y = \sqrt{x}$ over the x-axis.
- $y = (x - 3)^2$ shifts the graph of $y = x^2$ right 3 units.
- $y = (x + 2)^3$ shifts the graph of $y = x^3$ left 2 units.
- $y = 2^x - 1$ shifts the graph of $y = 2^x$ down 1 unit.
- $y = x + 4$ shifts the graph of $y = x^2$ up 4 units.

Graph of $y = |x|$

Because of the definition of absolute value of x, the graph of $y = |x|$ is comprised of the portion of the graphs of $y = x$ $(x \geq 0)$ and $y = -x$ $(x < 0)$ for which y is nonnegative as shown in Figure 5.6. Each ray of the graph is a reflection of the other in the y-axis.

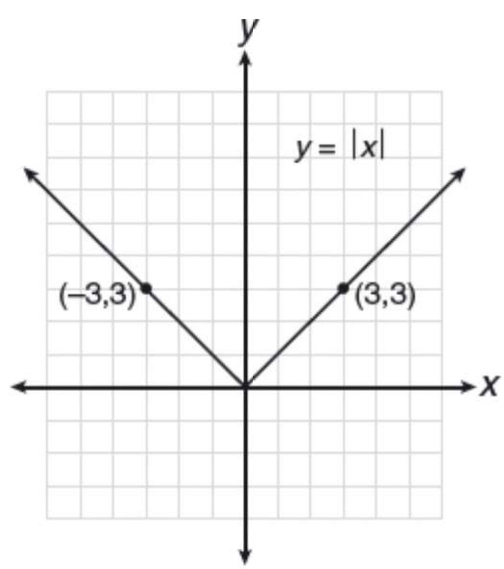


Figure 5.6 Graph of $y = |x|$.

As illustrated in Figure 5.7,

- The graph of $y = |x| + 2$ shifts the graph of $y = |x|$ up 2 units.

- The graph of $y = |x + 2|$ shifts the graph of $y = |x|$ left 2 units.

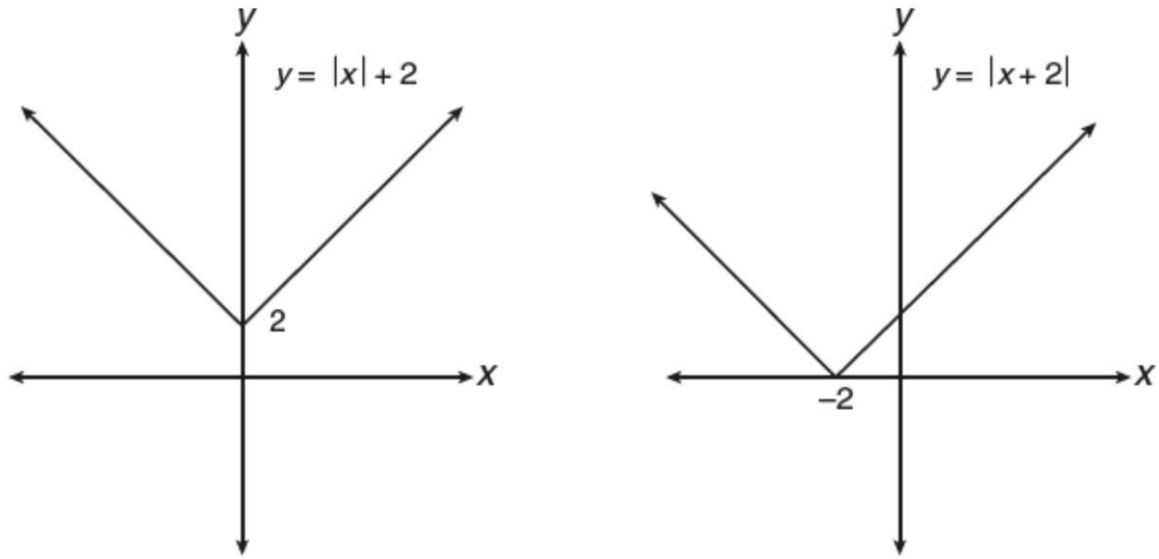


Figure 5.7 Shifting the graph of $y = |x|$

COMBINING HORIZONTAL AND VERTICAL SHIFTS

A graph may be shifted both vertically and horizontally as in Figure 5.8. The graph of $y = f(x - 2) + 3$ is the graph of $y = f(x)$ shifted sideways to the right 2 units and straight up 3 units. Each point (x, y) of the original graph corresponds to $(x + 2, y + 3)$ of the new graph. Since $(0, 0)$ is the vertex of $f(x) = x^2$, the vertex of the translated graph is $(0 + 2, 0 + 3) = (2, 3)$.

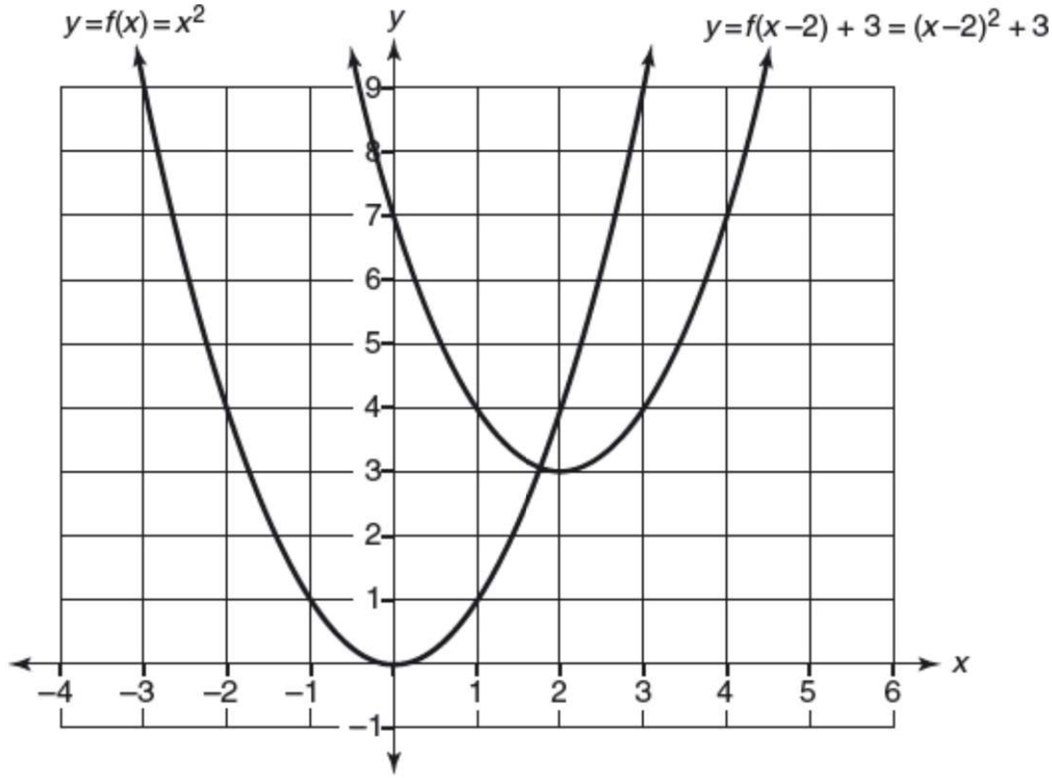


Figure 5.8 Translating the graph of $f(x) = x^2$ to the right 2 units and up 3 units

LESSON 5-6 TUNE-UP EXERCISES

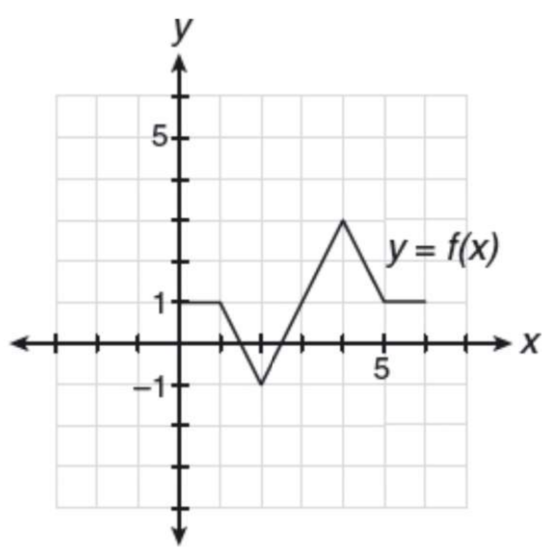
Multiple-Choice

1. The graph of $y = 2^{x-3}$ can be obtained by shifting the graph of $y = 2^x$

   (A) 3 units to the right
   (B) 3 units to the left
   (C) 3 units up
   (D) 3 units down

2. Which equation represents the line that is the reflection of the line $y = 2x - 3$ in the y-axis?

   (A) $y = -2x - 3$
   (B) $y = -2x + 3$
   (C) $y = 2x + 3$
   (D) $y = 3x - 2$



3. The figure above shows part of the graph of function f. If $f(x - 5) = f(x)$ for all values of x, what is the value of $f(19)$?

   (A) −1
   (B) 0
   (C) 1
   (D) 3

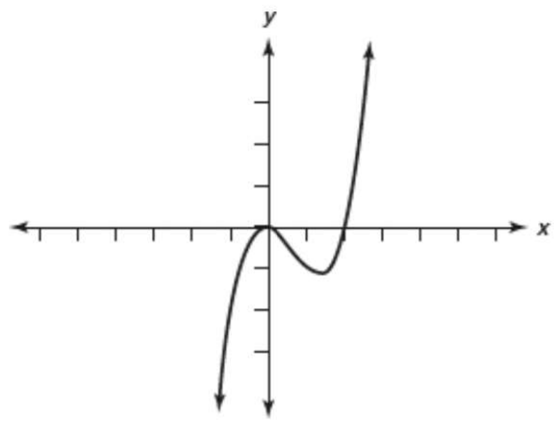
4. The endpoints of AB are $A(0, 0)$ and $B(9, -6)$. What is an equation of the line that contains the reflection of $\overline{AB}$ in the y-axis?

(A) $y = -\frac{3}{2}x$

(B) $y = -\frac{2}{3}x$

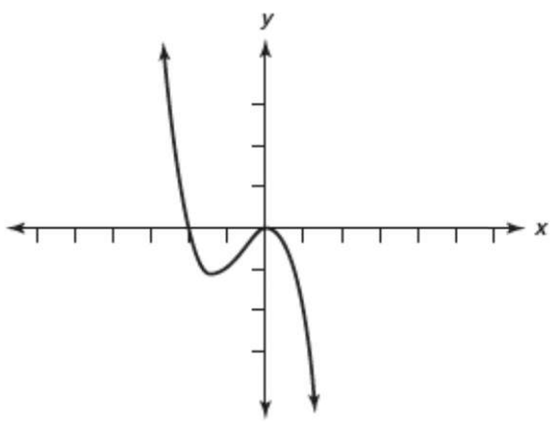
(C) $y = -x + 3$

(D) $y = \frac{2}{3}x$

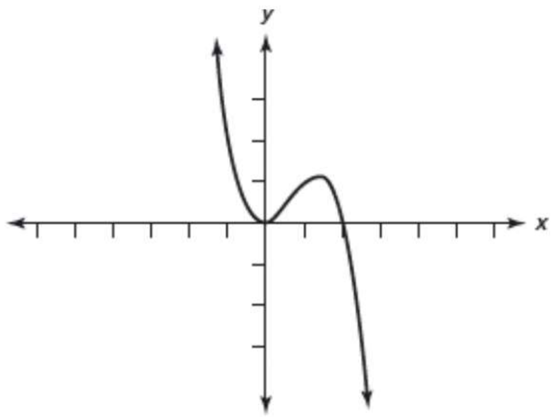


5. The figure above shows the graph of function f. If $g(x) = -f(x)$, which graph represents function g?

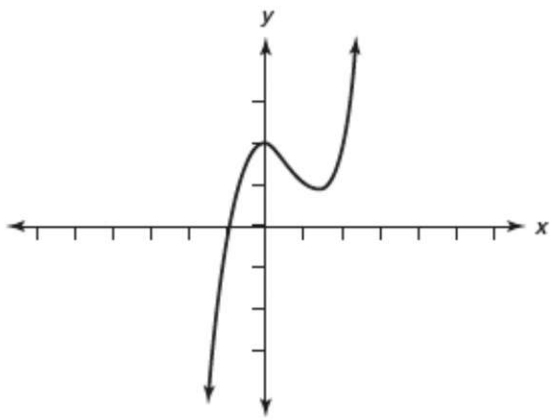
(A)



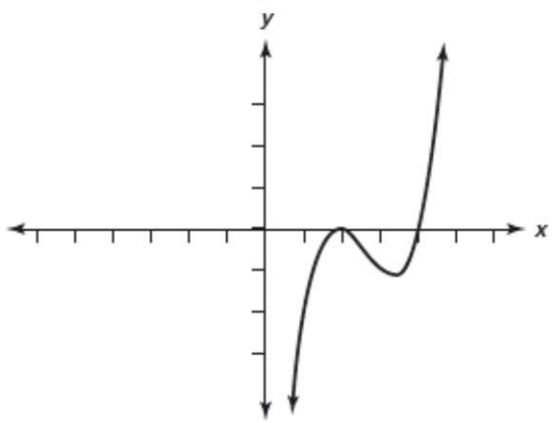
(B)



(C)

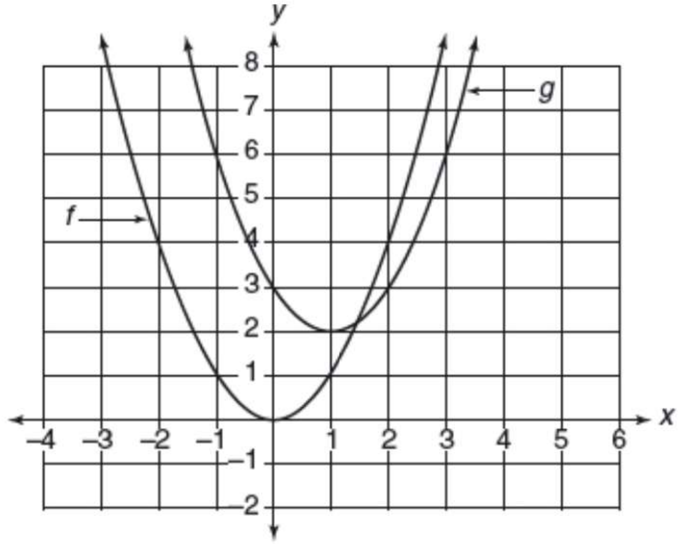


(D)



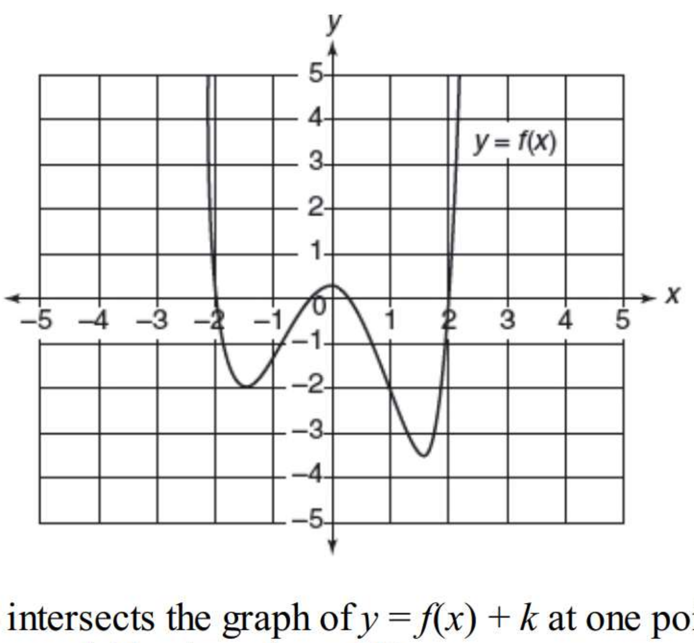
6. The point $(2, -1)$ on the graph $y = f(x)$ is shifted to which point on the graph of $y = f(x + 2)$?

(A) $(4, 1)$
(B) $(4, -1)$
(C) $(0, -1)$
(D) $(0, -3)$



7. The accompanying figure shows the graphs of functions f and g. If f is defined by $f(x) = x^2$ and g is defined by $g(x) = f(x + h) + k$, where h and k are constants, what is the value of $h + k$?

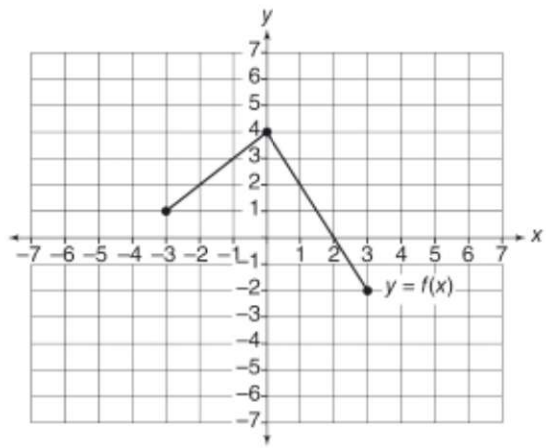
(A) −3
(B) −2
(C) −1
(D) 1



8. If $g(x) = -2$ intersects the graph of $y = f(x) + k$ at one point, which of these choices could be the value of k?

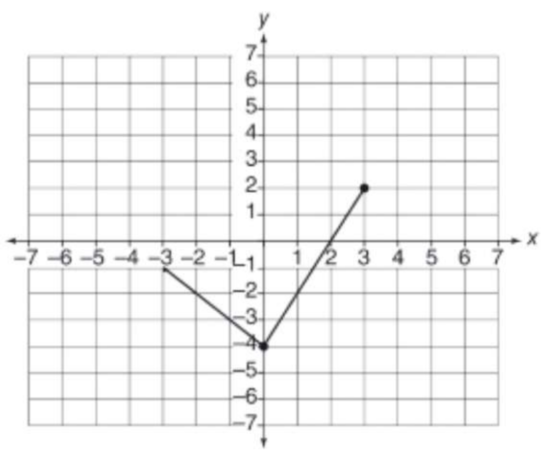
(A) −1.5

(B) −0.5
(C) 0
(D) 1.5

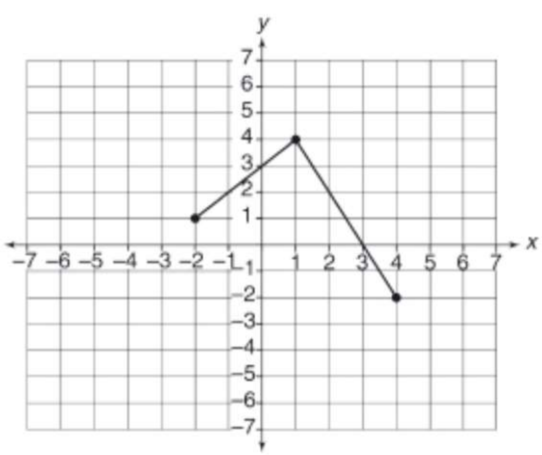


9. If the accompanying figure above shows the graph of function *f*, which of the following could represent the graph of $y = f(x + 1)$?

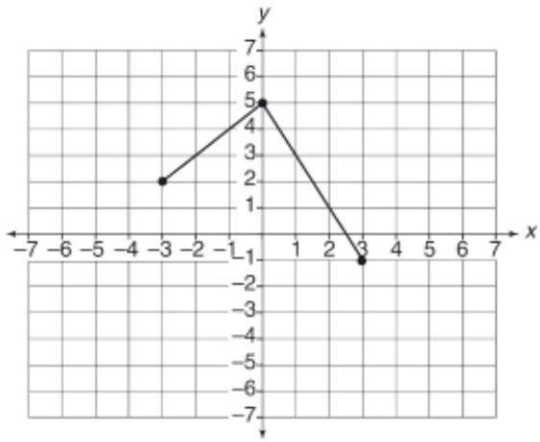
(A)



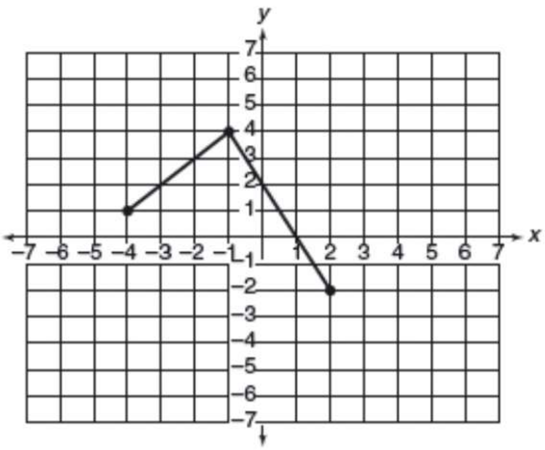
(B)

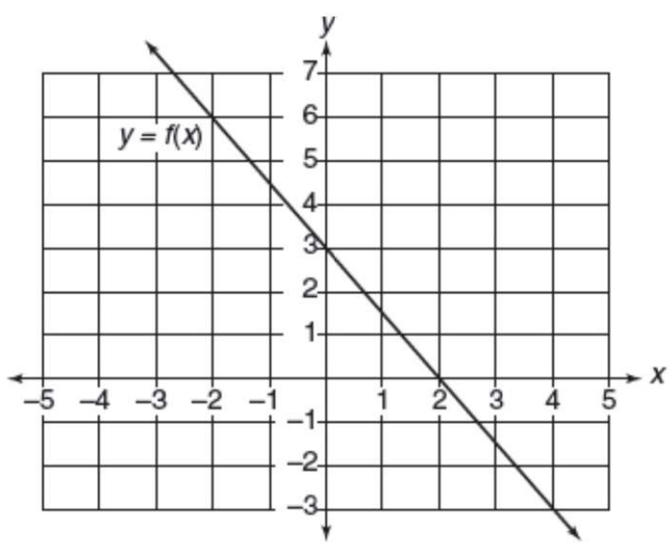


(C)



(D)





10. A linear function *f* is shown in the accompanying figure. If function *g* is the reflection of function *f* in the *x*-axis (not shown), what is the slope of the graph of function *g*?

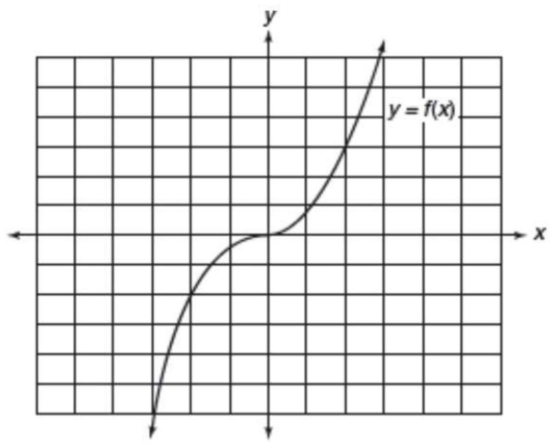
(A) $-\frac{3}{2}$

(B) $-\frac{2}{3}$

(C) $\frac{2}{3}$

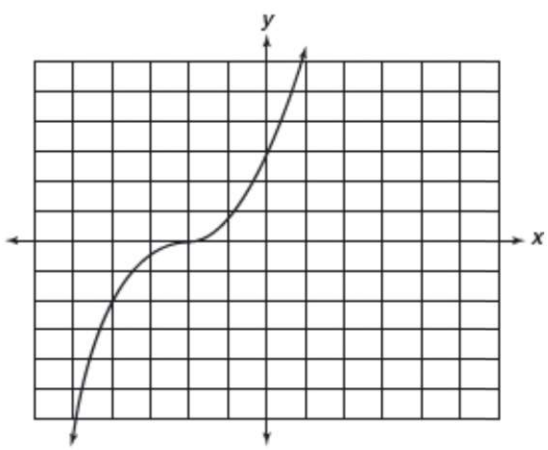
(D) $\frac{3}{2}$

11. The graph of $y = f(x)$ is shown below.

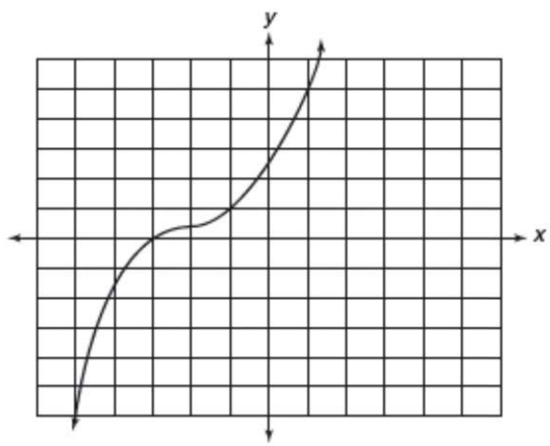


Which of the following could represent the graph of $y = f(x - 2) + 1$?

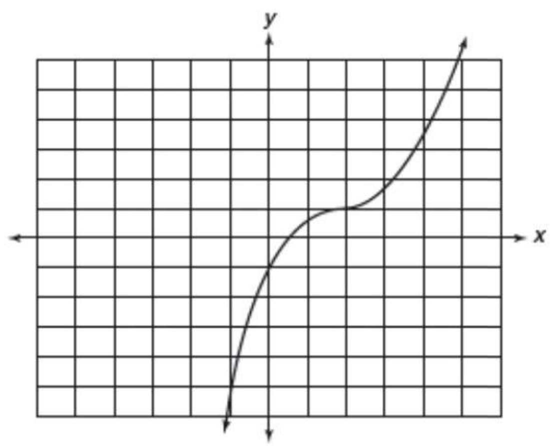
(A)



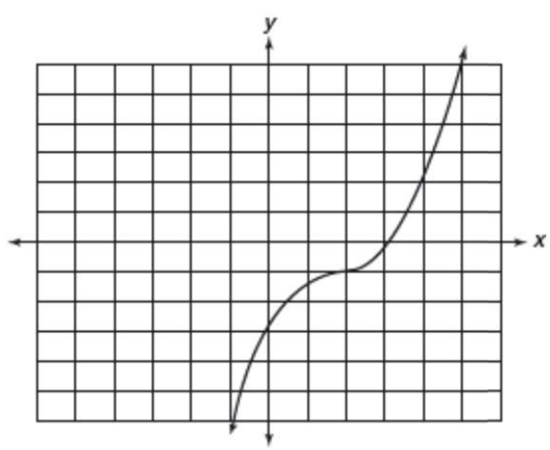
(B)

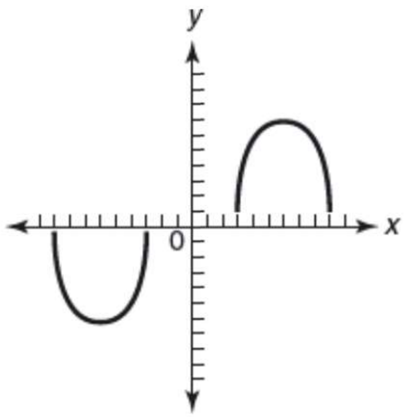


(C)



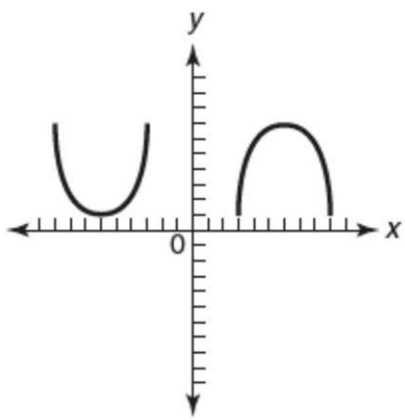
(D)



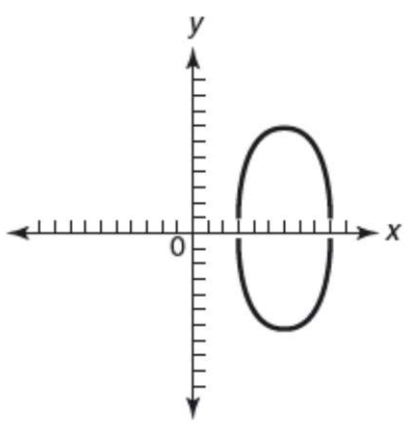


12. The graph of the function f is shown above. Which of the following could represent the graph of $y = |f(x)|$?

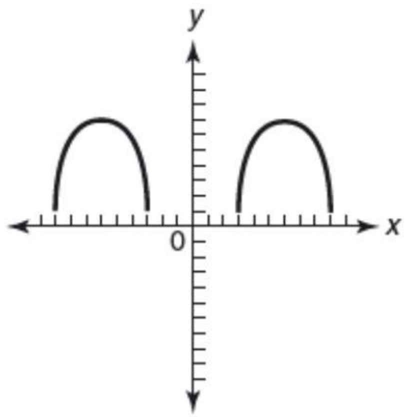
(A)



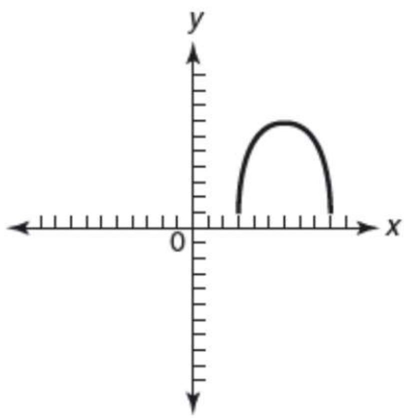
(B)



(C)

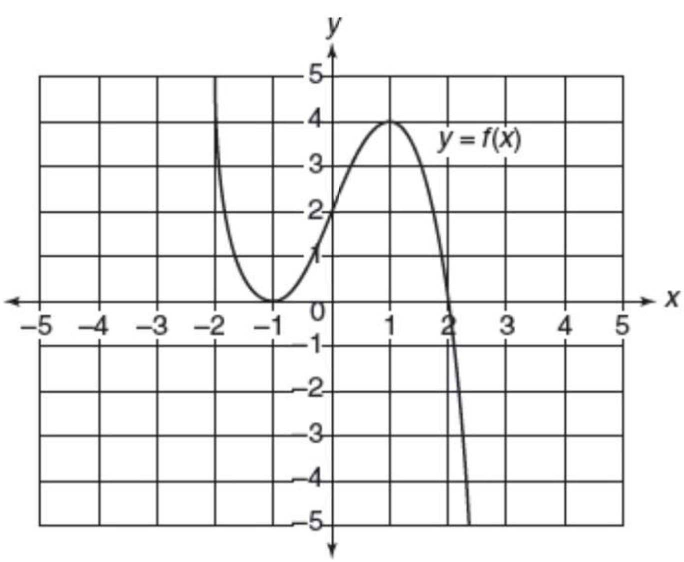


(D)



Grid-In

Questions 1 and 2 refer to the information and graph below.
Let function f be defined by the graph in the accompanying figure.



1. For what positive integer k is (1, 0) an x-intercept of the graph of $y = f(x - k)$?

2. Let m represent the number of points at which the graphs of $y = f(x)$ and $g(x) = 3$ intersect. Let n represent the number of points at which the graphs of $y = f(x) - 1$ and $g(x) = 3$ intersect. What is the value of $m + n$?

6

Additional Topics in Math

In addition to algebra, you are also expected to know and to be able to apply key facts and relationships from geometry and trigonometry.

You not only need to know right triangle trigonometry but also need to be familiar with the general angle, angles measured in radians as well as degrees, the unit circle, and the coordinate definitions of the three basic trigonometric functions.

"Additional Topics in Math" represents the last of the four major mathematics content groups tested by the redesigned SAT.

LESSONS IN THIS CHAPTER

Lesson 6-1 Reviewing Basic Geometry Facts
Lesson 6-2 Area of Plane Figures
Lesson 6-3 Circles and Their Equations
Lesson 6-4 Solid Figures
Lesson 6-5 Basic Trigonometry
Lesson 6-6 The Unit Circle

LESSON 6-1 REVIEWING BASIC GEOMETRY FACTS

OVERVIEW

This lesson summarizes some important relationships from your study of Geometry that you should already know.

ANGLES AND LINES

When two lines intersect, vertical angles (opposite angles) have the same measure. See Figure 6.1.

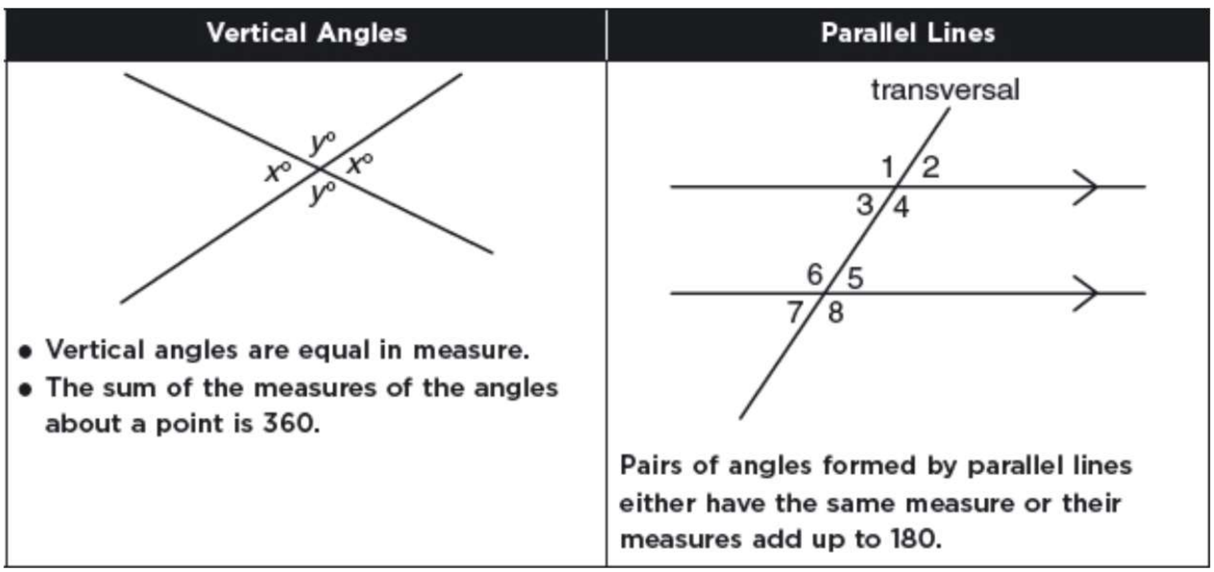


Figure 6.1 Angle relationships for intersecting and parallel lines

When two parallel lines are cut by another line, called a *transversal*, every pair of angles formed are either congruent (have the same measure) or are supplementary (have measures that add up to 180). In Figure 6.1, since the lines are parallel,

- Alternate interior angles 3 and 5 are equal in measure as are alternate interior angles 4 and 6.
- Corresponding pairs of angles 1 and 2, 3 and 7, 2 and 5, as well as 4 and 8 have equal measures.

Angles 2 and 8 look like they have different measures (one acute angle and one obtuse angle) so you can assume because the lines are parallel that their measures add up to 180.

TRIANGLES AND POLYGONS

For triangles (see Figure 6.2),

- The sum of the measures of the three angles is 180.
- The measure of an exterior angle is equal to the sum of the two nonadjacent interior angles of the triangle.

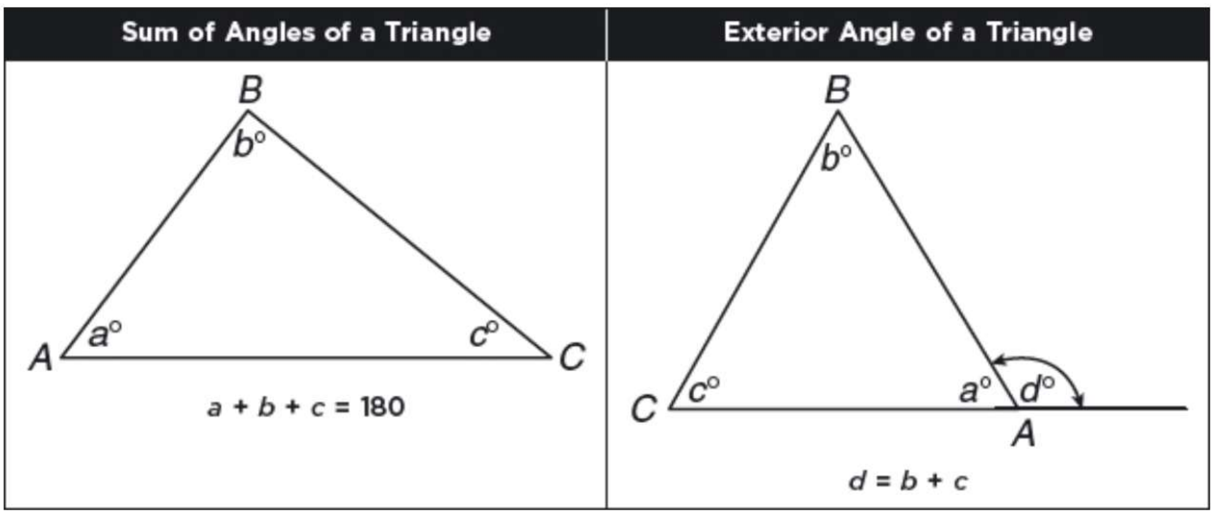


Figure 6.2 Angle relationships in a triangle

For any polygon with *n* sides (see Figure 6.3),

- The sum of the measures of the interior angles is $(n - 2) \cdot 180$. The sum of the measures of the four angles of a quadrilateral is $(4 - 2) \cdot 180 = 2 \cdot 180 = 360$.
- The sum of the exterior angles, one angle at each vertex, is 360.

MATH REFERENCE FACT

In a **regular polygon**, all of the sides have the same length and all of the angles have the same measure. For a *regular* polygon with *n* sides,

- The measure of each exterior angle is $\frac{360}{n}$.
- The measure of each interior angle is $180 - \frac{360}{n}$.

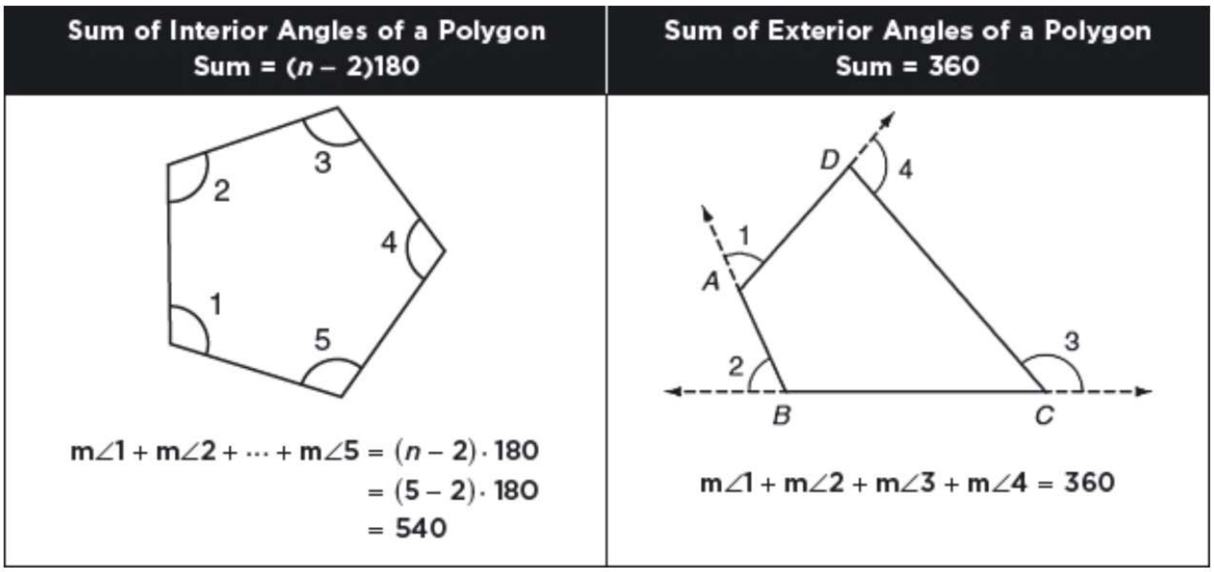


Figure 6.3 Angle relationships for polygons

ISOSCELES, EQUILATERAL, AND RIGHT TRIANGLES

If two sides of a triangle have the same length, then the angles that face these sides have the same measures. If all 3 sides of a triangle have the same length, then the three angles of the triangle have the same measure. See Figure 6.4.

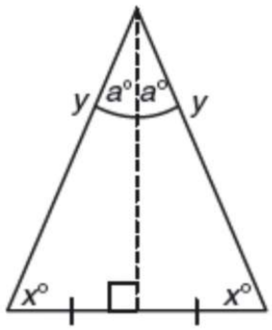
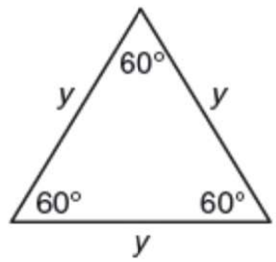
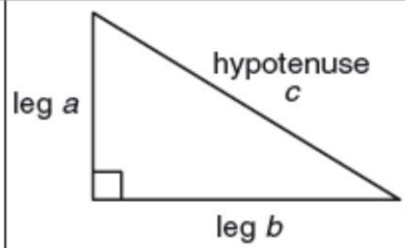
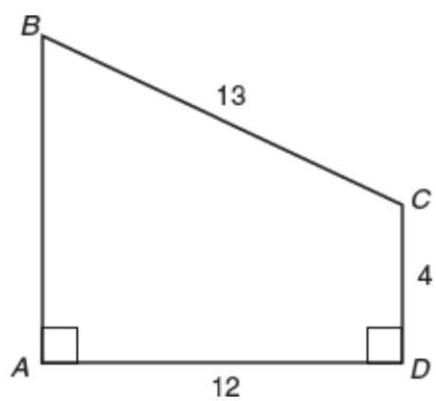
Isosceles Triangle (Angles facing equal sides have equal measures)	Equilateral Triangle (Three sides have the same length)	Right Triangle (Side facing the 90° angle is the hypotenuse)
 Altitude to base bisects the base and the vertex angle	 An equilateral triangle is also equiangular	 $a^2 + b^2 = c^2$ Common Side Length Triples 3-4-5 5-12-13 8-15-17

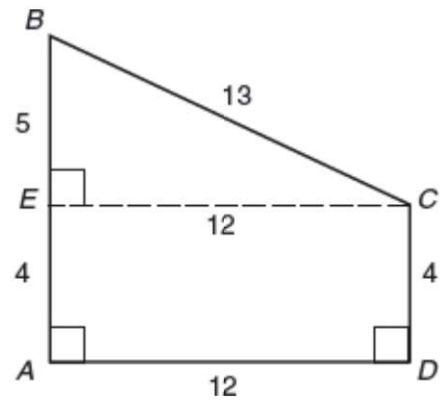
Figure 6.4 Relationships in isosceles, equilateral, and right triangles

➡ Example



Note: Figure not drawn to scale

To find the perimeter of trapezoid *ABCD* above, first draw $\overline{EC}$ parallel to $\overline{AD}$ thereby forming rectangle *AECD*, which means that $EC = AD = 12$ and $EA = CD = 4$.



The lengths of the sides of right triangle *BEC* form a 5-12-13 Pythagorean triple with *BEC* = 5. Hence, the perimeter of trapezoid *ABCD* is 12 + 9 + 13 + 4 = 38.

SPECIAL RIGHT TRIANGLE RELATIONSHIPS

The lengths of the sides of a 45-45 right triangle and of a 30-60 right triangle are in fixed ratios as illustrated in Figure 6.5. These relationships will be provided in the reference page in your SAT test booklet.

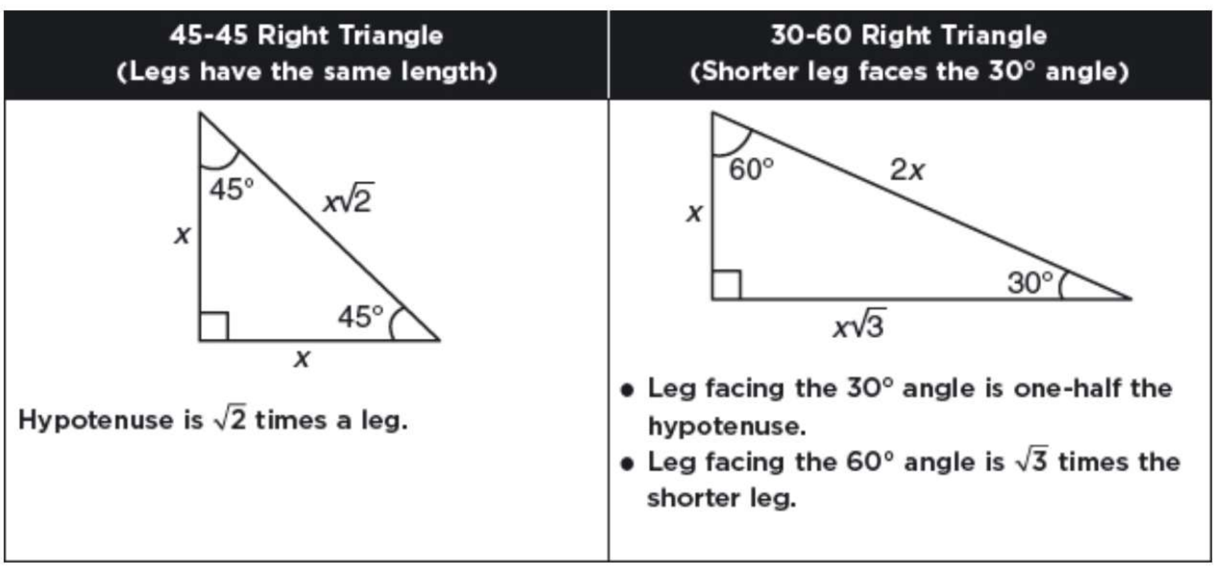
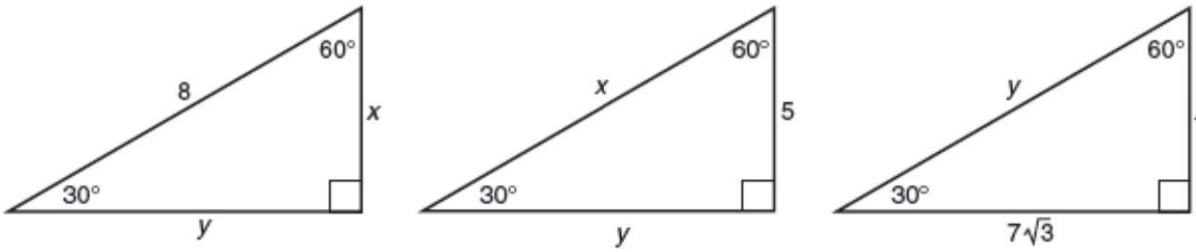


Figure 6.5 Special right triangle relationships

Here are some examples:



$x = \frac{1}{2}(8) = 4$

$y = x\sqrt{3} = 4\sqrt{3}$

$x = 2(5) = 10$

$y = 5\sqrt{3}$

$x = 7$

$y = 2(7) = 14$

SIMILAR TRIANGLES

If two angles of one triangle are congruent to the corresponding angles of another triangle, then the two triangles are similar (:). If two triangles are similar, as shown in Figure 6.6, then the lengths of pairs of corresponding sides have the same ratio:

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

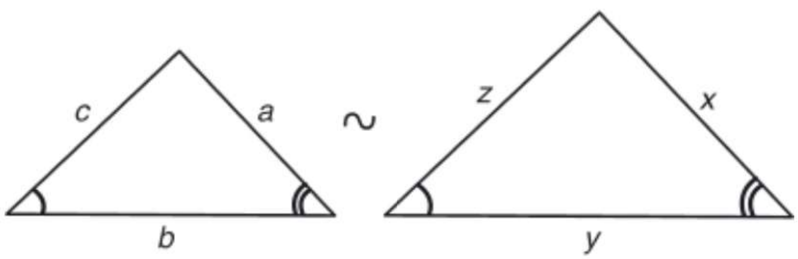
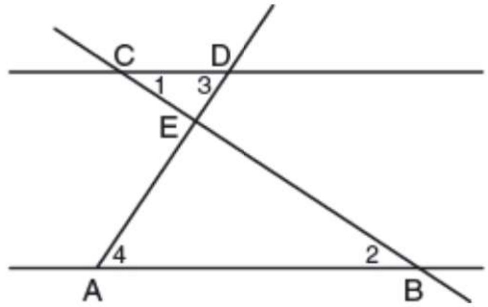


Figure 6.6 Similar triangles

> **TIP**
>
> **Corresponding sides of two similar triangles always face corresponding congruent angles.**

➡ Example

In the accompanying figure, $\overline{AB} \| \overline{CD}$. If $AB = 40$, $CD = 16$, and $BC = 49$, what is the length of $\overline{BE}$?

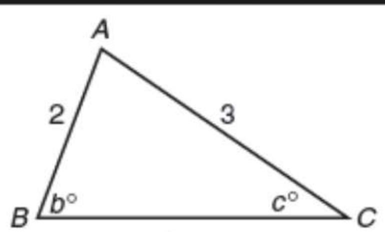
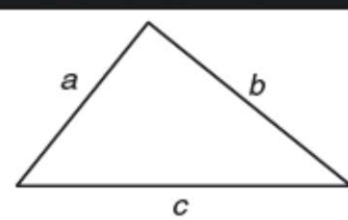


Angles pairs 1 and 2, as well as 3 and 4, are formed by parallel lines (alternate interior angles) and look congruent, so for SAT purposes, you can assume that they are congruent: $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$. Hence, $\triangle ABE \sim \triangle DCE$. If x represents the length of $\overline{BE}$, then:

$$\frac{x}{49-x} = \frac{40}{16}$$
$$\frac{x}{49-x} = \frac{5}{2}$$
$$2x = 245 - 5x$$
$$\frac{7x}{7} = \frac{245}{7}$$
$$x = 35$$

INEQUALITIES IN A TRIANGLE

If two sides of a triangle have different lengths, then the angles opposite these sides have different measures with the larger angle facing the longer side. See Figure 6.7.

Unequal Sides Imply Unequal Angles	Side Length Restrictions in a Triangle
 $b > c$ **Larger angle faces the longer side.** **Figure 6.7** Inequality relationships in a triangle	 $c < a + b$ and $c > b - a$ **Figure 6.8** Side length restrictions in a triangle

SIDE LENGTH RESTRICTIONS IN A TRIANGLE

In a triangle, the length of each side must be less than the sum of the lengths of the other two sides and greater than their difference. See Figure 6.8.

➥ Example

If 3, 8, and x represent the lengths of the sides of a triangle, how many integer values for x are possible?

Solution

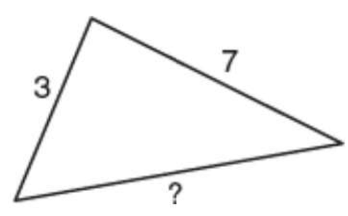
- The value of x must be greater than $8 - 3 = 5$ *and* less than $8 + 3 = 11$.
- Since $x > 5$ and $x < 11$, x must be between 5 and 11.
- Because it is given that x is an integer, x can be equal to 6, 7, 8, 9, or 10.
- Hence, there are five possible integer values for x.

➥ Example

If the lengths of two sides of an isosceles triangle are 3 and 7, what is the perimeter of the triangle?

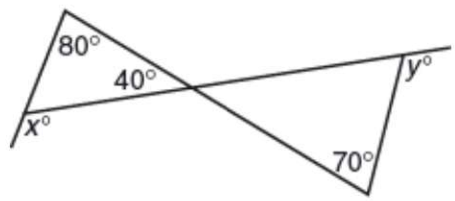
Solution

The length of the third side of this isosceles triangle could be 3 or 7. If it were 3, then the third side, 7, would be greater than the sum of the lengths of the other two sides, which is not possible. Hence, the length of the third side is 7 so the perimeter of the triangle is $7 + 7 + 3 =$ **17.**



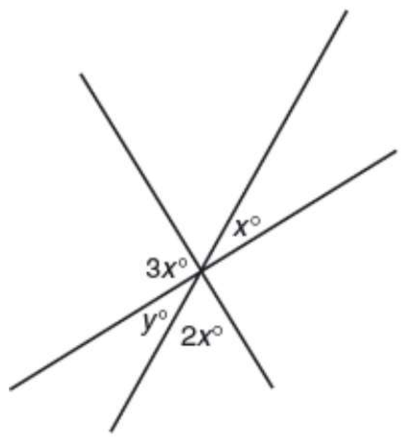
LESSON 6-1 TUNE-UP EXERCISES

Multiple-Choice



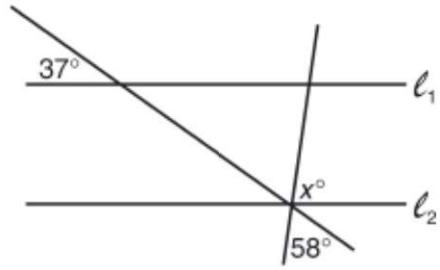
1. In the figure above, $x + y =$

   (A) 270
   (B) 230
   (C) 210
   (D) 190



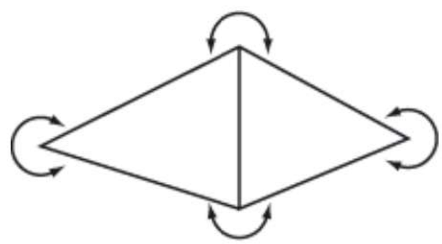
2. In the figure above, what is the value of y?

   (A) 20
   (B) 30
   (C) 45
   (D) 60



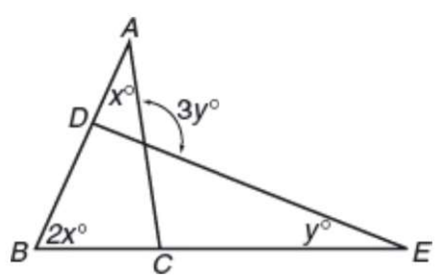
3. In the figure above, if $\ell_1 \parallel \ell_2$, what is the value of x?

(A) 90
(B) 85
(C) 75
(D) 70



4. In the figure above, what is the sum of the degree measures of all of the angles marked?

(A) 540
(B) 720
(C) 900
(D) 1080



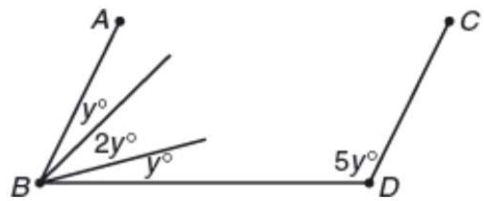
5. In the figure above, what is y in terms of x?

(A) $\frac{3}{2}x$

(B) $\frac{4}{3}x$

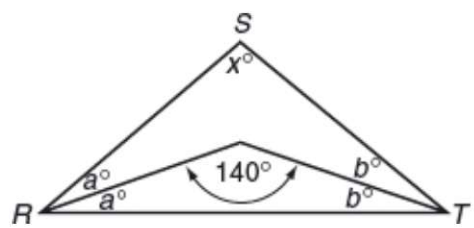
(C) x

(D) $\frac{3}{4}x$



6. In the figure above, if line segment AB is parallel to line segment CD, what is the value of y?

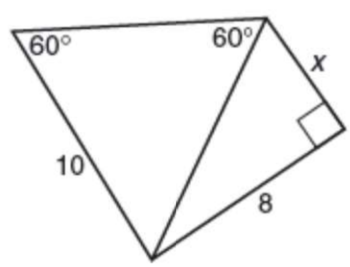
   (A) 12
   (B) 15
   (C) 18
   (D) 20



Note: Figure not drawn to scale.

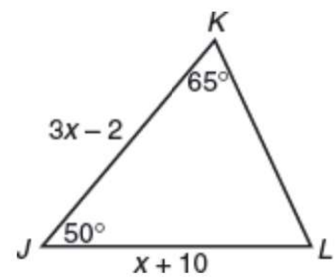
7. In $\triangle RST$ above, what is the value of x?

   (A) 80
   (B) 90
   (C) 100
   (D) 110



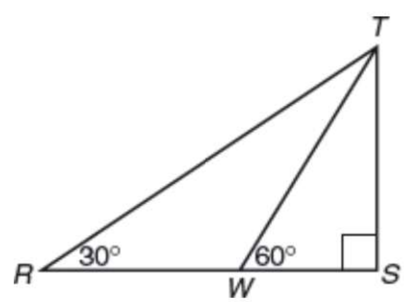
8. In the figure above, $x =$

(A) 4
(B) 6
(C) $4\sqrt{2}$
(D) $4\sqrt{3}$



9. In $\triangle JKL$ above, what is the value of x?

(A) 2
(B) 3
(C) 4
(D) 6



Note: Figure not drawn to scale.

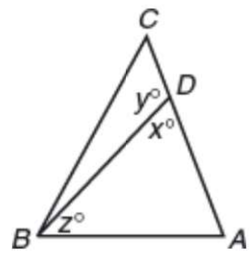
10. In the figure above, what is the ratio of RW to WS?

(A) $\sqrt{2}$ to 1
(B) $\sqrt{3}$ to 1
(C) 2 to 1
(D) 3 to 1

11. Katie hikes 5 miles north, 7 miles east, and then 3 miles north again. What number of miles, measured in a straight line, is Katie from her starting point?

(A) $\sqrt{83}$

(B) 10
(C) $\sqrt{113}$
(D) 13



Note: Figure not drawn to scale.

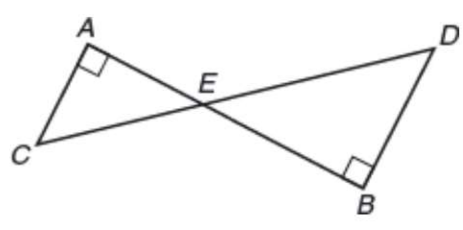
12. In $\triangle ABC$, if $AB = BD$, which of the following statements must be true?

I. $x > z$
II. $y > x$
III. $AB > BC$

(A) I only
(B) II only
(C) I and II only
(D) II and III only

13. How many different triangles are there for which the lengths of the sides are 3, 8, and n, where n is an integer and $3 < n < 8$?

(A) Two
(B) Three
(C) Four
(D) Five



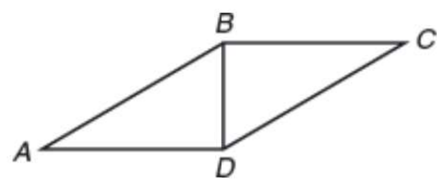
14. If, in the figure above, $AC = 3$, $DB = 4$, and $AB = 14$, then $AE =$

(A) 4.5
(B) 6

(C) 8
(D) 10.5

15. What is the number of sides of a polygon in which the sum of the degree measures of the interior angles is 4 times the sum of the degree measures of the exterior angles?

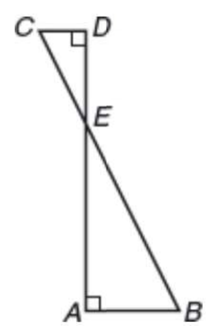
(A) 10
(B) 12
(C) 14
(D) No such polygon exists.



16. For parallelogram *ABCD* above, if $AB > BD$, which of the following statements must be true?

I. $CD < BD$
II. $\angle ADB > \angle C$
III. $\angle CBD > \angle A$

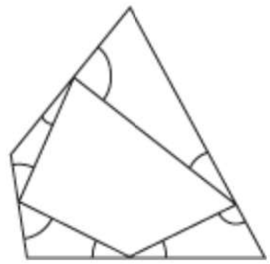
(A) None
(B) I only
(C) II and III only
(D) I and III only



17. If, in the figure above, $CD = 1$, $AB = 2$, and $AD = 6$, then $BC =$

(A) 5
(B) 9

(C) $2 + \sqrt{5}$
(D) $3\sqrt{5}$



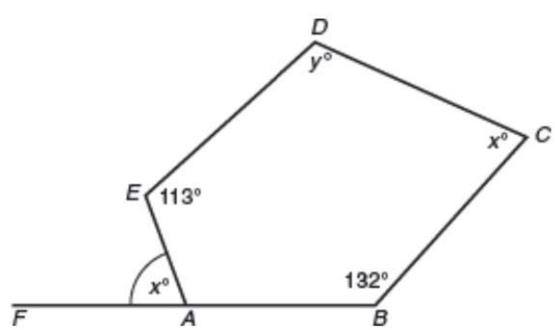
18. In the figure above, what is the sum of the degree measures of the marked angles?

(A) 120
(B) 180
(C) 360
(D) It cannot be determined from the information given.

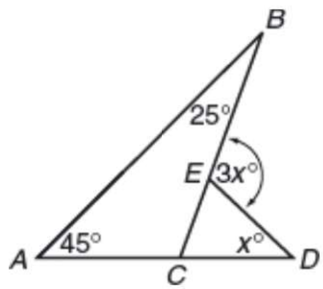
19. If each interior angle of a regular polygon measures 140°, how many side does the polygon have?

(A) 5 sides
(B) 6 sides
(C) 9 sides
(D) 10 sides

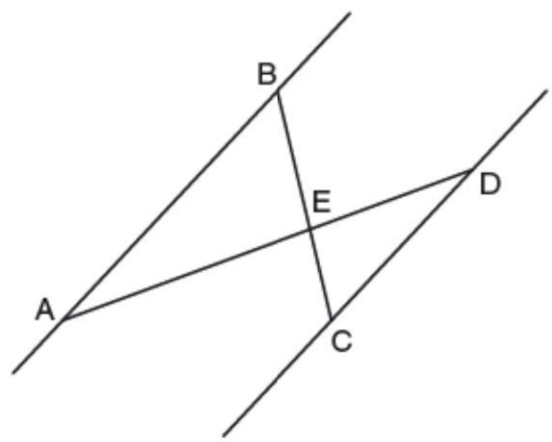
Grid-In



1. In the accompanying figure of pentagon *ABCDE*, points *F*, *A*, and *B* lie on the same line. What is the value of *y*?

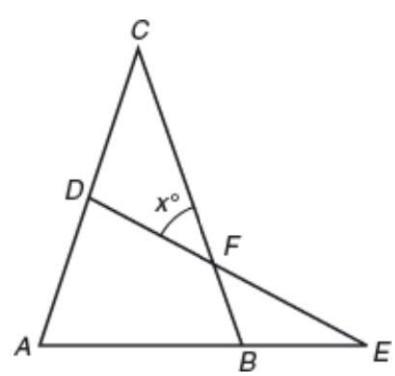


2. In the figure above, what is the value of x?



3. In the figure above, $\overline{AB} \| \overline{CD}$, $AD = 30$, $AB = 21$, and $CD = 15$. What is the length of $\overline{DE}$?

4. In the accompanying diagram of triangle ABC, $AC = BC$, D is a point on $\overline{AC}$, $\overline{AB}$ is extended to E, and $\overline{DFE}$ is drawn so that $\triangle ADE \sim \triangle ABC$. If $m\angle C = 30$, what is the value of x?



5. Two hikers started at the same location. One traveled 2 miles east and then 1 mile north. The other traveled 1 mile west and then 3 miles south. At the end of their hikes, how many miles apart were the two hikers?

LESSON 6-2 AREA OF PLANE FIGURES

OVERVIEW

This lesson reviews some key properties of parallelograms as well as the formulas to find the areas of parallelograms, triangles, and trapezoids.

PROPERTIES OF PARALLELOGRAMS AND RECTANGLES

A **parallelogram** is a quadrilateral whose opposite sides are parallel. In addition,

- Opposite sides have the same length:

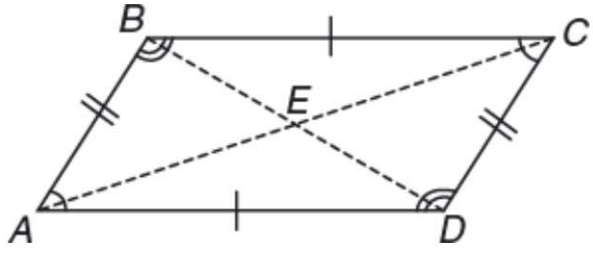
$$AB = CD \text{ and } AD = BC$$

- Opposite angles have the same measure:

$$\angle A = \angle C \text{ and } \angle B = \angle D$$

- Diagonals bisect each other:

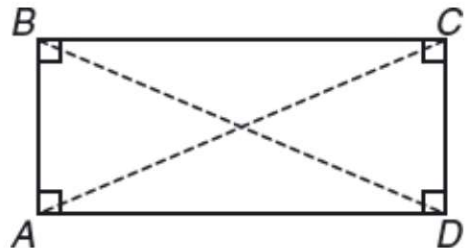
$$AE = EC \text{ and } BE = ED$$



$$\overline{AD} \parallel \overline{BC} \text{ and } \overline{AB} \parallel \overline{CD}$$

A **rectangle** is a parallelogram with four right angles. A rectangle has all the properties of a parallelogram and the additional property that its diagonals have the same length:

$$AC = BD$$



AREA OF A PARALLELOGRAM AND RECTANGLE

The area of a parallelogram and the area of a rectangle are each obtained by multiplying the base and altitude together as shown in Figure 6.9.

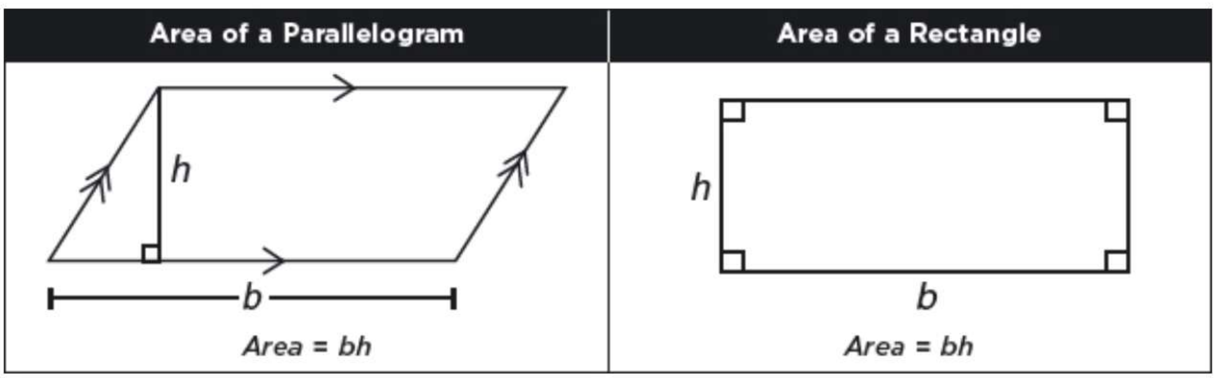
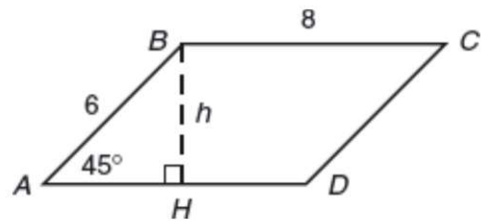


Figure 6.9 Area formulas for a parallelogram and rectangle

➡ Example

To find the area of parallelogram *ABCD*, draw perpendicular segment *BH*, as shown. Since *BH* is the side opposite a 45° angle in a right triangle:

$$h = 6 \times \frac{\sqrt{2}}{2} = 3\sqrt{2}$$

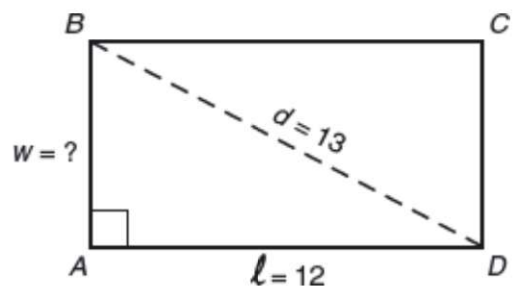


Opposite sides of a parallelogram are equal, so $AD = 8$. Hence,

$$\begin{aligned}\text{Area of parallelogram } ABCD &= bh \\ &= AD \times h \\ &= 8 \times 3\sqrt{2} \\ &= 24\sqrt{2}\end{aligned}$$

➥ Example

To find the area of rectangle $ABCD$, note that the diagonal forms a (5-12-13) right triangle so the width of the rectangle is 5, Hence:



Area of rectangle $= \ell w = 12 \times 5 = 60$

AREA OF A RHOMBUS AND A SQUARE

A **rhombus** is a parallelogram in which all sides have the same length. A **square** is a rhombus with four right angles.

- The diagonals of a square have the same length.
- In both a rhombus and square, the diagonals are perpendicular bisectors of each other.

Figure 6.10 summarizes the area formulas for a square and rhombus.

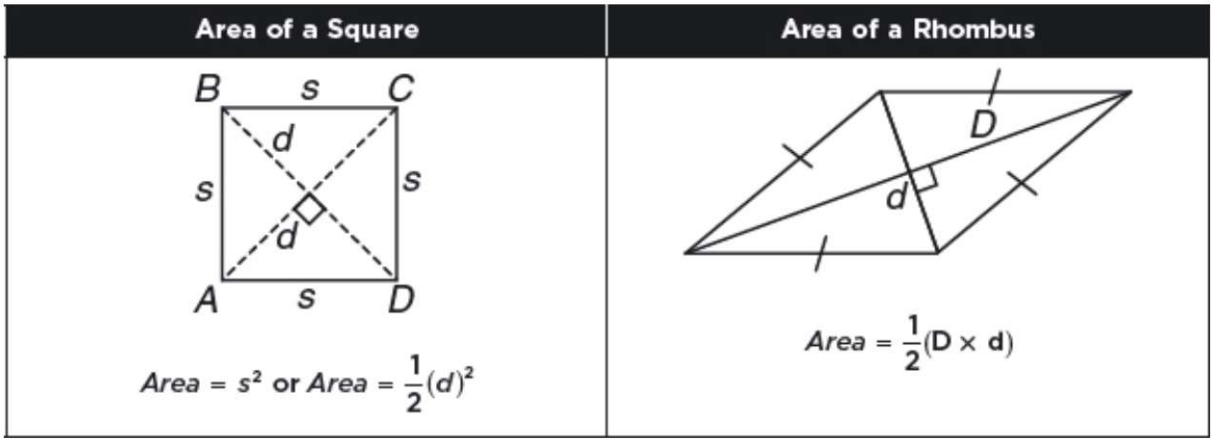
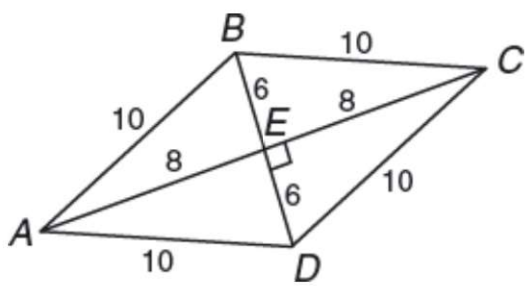


Figure 6.10 Area of a square and area of a rhombus

➡ Example

If rhombus *ABCD* has a side length of 10 and the longer diagonal measures 16, then the shorter diagonal measures 12 since



- diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other and bisect each other, which makes $\triangle AED$ a 6-8-10 right triangle.
- $BD = 6 + 6 = 12$

Hence,

$$\text{Area rhombus } ABCD = \frac{1}{2}(16)(12) = 96$$

AREA OF A TRIANGLE

Figure 6.11 shows the area formulas for different type of triangles.

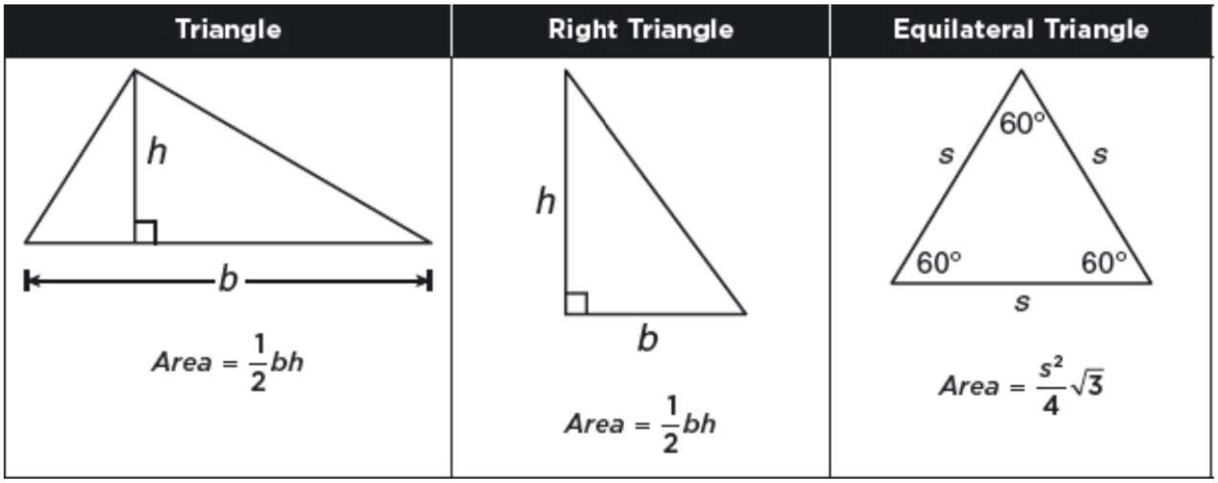
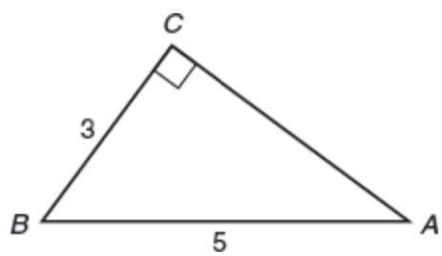


Figure 6.11 Area formulas for triangles

➡ Example

To find the area of $\triangle ABC$, note that the lengths of the sides of $\triangle ABC$ form a (3-4-5) Pythagorean triple, where $AC = 4$. Hence,

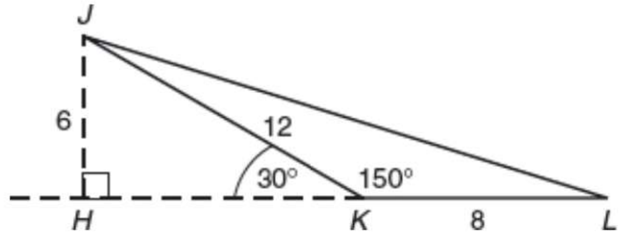


$$\text{Area of } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6$$

Note that in a right triangle either leg is the base and the other leg is the height.

➡ Example

To find the area of $\triangle JKL$, drop a perpendicular segment from vertex J to side KL, extending it as necessary. Since $\angle JKH$ measures 30°:



$$h = JH = \frac{1}{2} \times 12 = 6$$

and

$$\text{Area of } \triangle JKL = \frac{1}{2}\, bh = \frac{1}{2}(8)(6) = 24$$

AREA OF A REGULAR HEXAGON

As shown in Figure 6.12, a regular hexagon can be divided into 6 equilateral triangles:

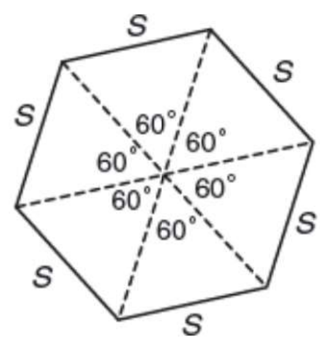


Figure 6.12 Dividing a regular hexagon into 6 equilateral triangles

Since the area of each equilateral triangle is $\frac{s^2}{4}\sqrt{3}$, the area of a regular hexagon is

$$6\left(\frac{s^2}{4}\sqrt{3}\right)$$

➡ Example

If the area of a regular hexagon is $96\sqrt{3}$, what is its perimeter?

Solution

Find the length, *s*, of each side of the regular hexagon:

$$6\left(\frac{s^2}{4}\sqrt{3}\right) = 96\sqrt{3}$$
$$s^2\sqrt{3} = \frac{2}{3}\left(96\sqrt{3}\right)$$
$$s^2 = 64$$
$$s = \sqrt{64} = 8$$

Hence, the perimeter of the regular hexagon is 6 × 8 = 48.

AREA OF A TRAPEZOID

A **trapezoid** is a quadrilateral with one pair of parallel sides, as shown in Figure 6.13.

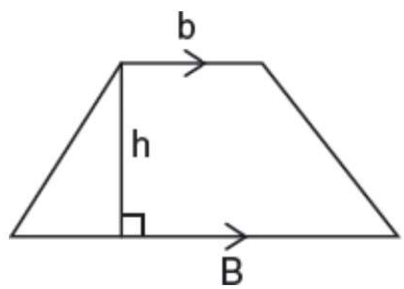


Figure 6.13 Trapezoid

The area of a trapezoid is the altitude times one-half the sum of the lengths of the two parallel sides called *bases*:

$$Area = h \times \frac{1}{2}(B + b)$$

In an **isosceles trapezoid**, the nonparallel sides (legs) have the same length and the bases angles have the same measure, as shown in Figure 6.14.

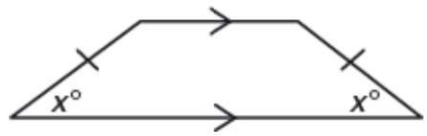
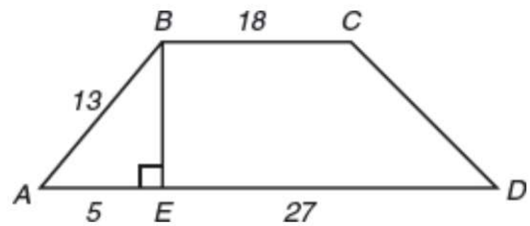


Figure 6.14 Isosceles trapezoid

➡ Example

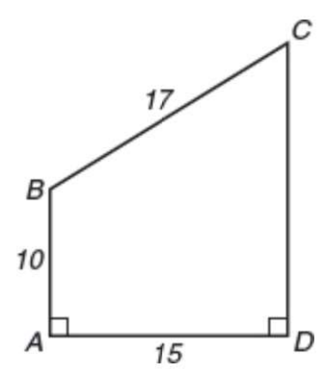
To find the area of trapezoid *ABCD*, use the fact that the lengths of the sides of right triangle *AEB* form a (5-12-13) Pythagorean triple, where height *BE* = 12. The length of lower base *AD* = *AE* + *ED* = 5+ 27 = 32.



$$\text{Area of trapezoid } ABCD = \text{Height} \times \frac{(\text{Sum of bases})}{2}$$
$$= (BE) \times \frac{(AD + BC)}{2}$$
$$= (12) \times \frac{(32 + 18)}{2}$$
$$= 300$$

➡ Example

To find the area of trapezoid *ABCD*, first find the length of base *CD* by drawing height *BH* to *CD*. Since parallel lines are everywhere equidistant, *BH* = *AD* = 15. The lengths of the sides of right triangle *BHC* form an (8-15-17) Pythagorean triple, where *CH* = 8. Thus, *CD* = *CH* + *HD* = 8 + 10 = 18.



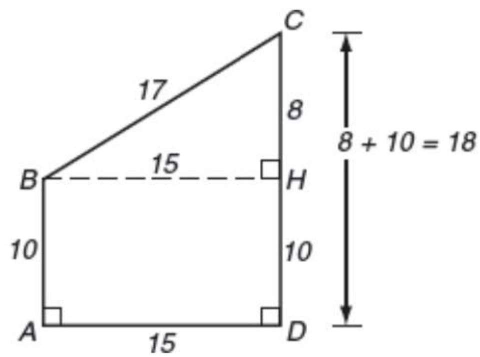
$$\text{Area of trapezoid } ABCD = \text{Height} \times \frac{(\text{Sum of bases})}{2}$$

$$= (BH)\frac{(AB + CD)}{2}$$

$$= (15)\frac{(10 + 18)}{2}$$

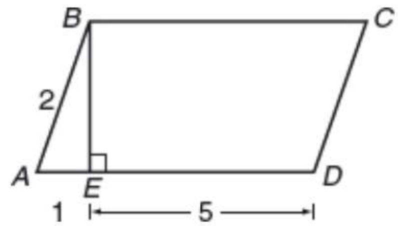
$$= 15 \times 14$$

$$= 210$$



LESSON 6-2 TUNE-UP EXERCISES

Multiple-Choice

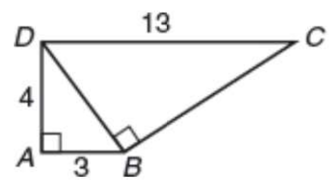


1. In the figure above, what is the area of parallelogram *ABCD*?

   (A) $4\sqrt{2}$
   (B) $4\sqrt{3}$
   (C) $6\sqrt{2}$
   (D) $6\sqrt{3}$

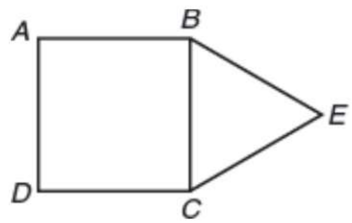
2. What is the area of a square with a diagonal of $\sqrt{2}$?

   (A) $\frac{1}{2}$
   (B) 1
   (C) $\sqrt{2}$
   (D) 2



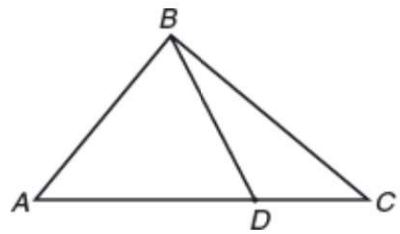
3. In the figure above, what is the area of quadrilateral *ABCD?*

   (A) 28
   (B) 32
   (C) 36
   (D) 42



4. In the figure above, if the area of square *ABCD* is 64, what is the area of equilateral triangle *BEC*?

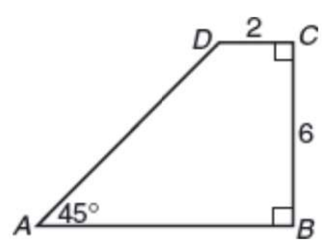
   (A) 8
   (B) $8\sqrt{3}$
   (C) $12\sqrt{3}$
   (D) $16\sqrt{3}$



5. In the figure above, the ratio of *AD* to *DC* is 3 to 2. If the area of $\triangle ABC$ is 40, what is the area of $\triangle BDC$?

   (A) 16
   (B) 24
   (C) 30
   (D) 36

Questions 6–7 are based on the diagram below.



Note: Figure not drawn to scale.

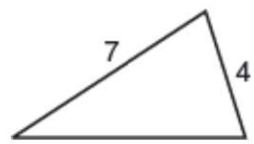
6. What is the perimeter of quadrilateral *ABCD*?

   (A) $16 + 2\sqrt{2}$

(B) $16 + 6\sqrt{2}$
(C) 28
(D) $22 + 6\sqrt{2}$

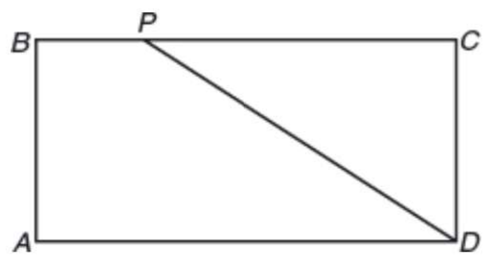
7. What is the area of quadrilateral *ABCD*?

(A) 20
(B) 24
(C) 30
(D) 36



8. If the perimeter of the triangle above is 18, what is the area of the triangle?

(A) $2\sqrt{33}$
(B) $6\sqrt{5}$
(C) 14
(D) $9\sqrt{5}$



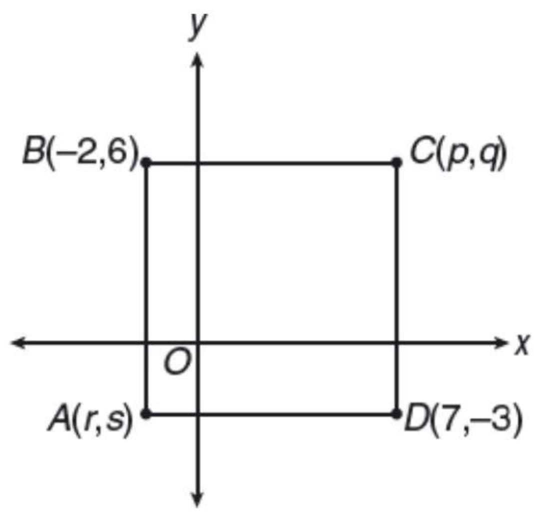
9. In rectangle *ABCD*, point *P* divides *BC* such that *BP* is 25% of the length of *BC*. If the area of quadrilateral *ABPD* is $\frac{3}{4}$, what is the area of rectangle *ABCD*?

(A) $\frac{15}{6}$

(B) $\frac{9}{8}$

(C) $\frac{6}{5}$

(D) $\frac{3}{2}$

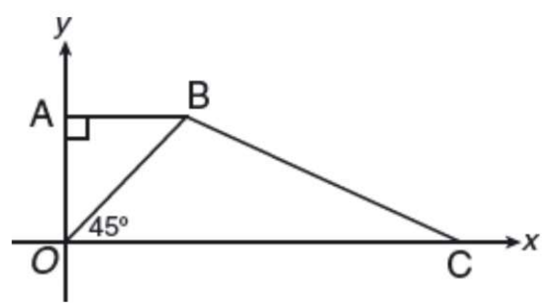


10. In the figure above, what is an equation of the line that contains diagonal *AC* of square *ABCD*?

 (A) $y = 2x + 1$

 (B) $y = \frac{1}{2}x - 2$

 (C) $y = 2x - 8$
 (D) $y = x - 1$



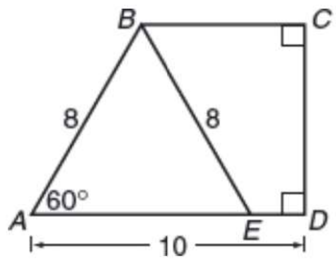
11. In the figure above, $\overline{OA} \perp \overline{AB}$, and $m\angle BOC = 45°$. If the coordinates of point *A* are (0, 3) and the coordinates of point *C* are (7, 0), what is the number of square units in the area of quadrilateral *OABC*?

 (A) 15
 (B) 20
 (C) 25
 (D) 30

12. If one pair of opposite sides of a square are increased in length by 20% and the other pair of sides are increased in length by 50%, by what

percent is the area of the rectangle that results greater than the area of the original square?

(A) 80%
(B) 77%
(C) 75%
(D) 70%

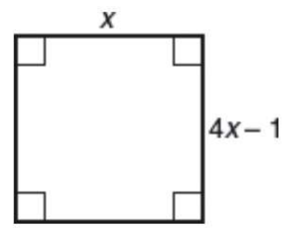


13. In the figure above, what is the area of quadrilateral *BCDE*?

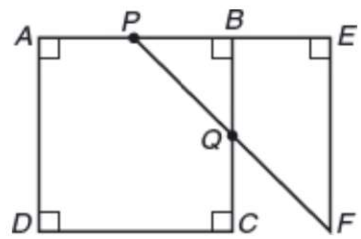
(A) $8\sqrt{3}$
(B) $16\sqrt{3}$
(C) $8 + 4\sqrt{3}$
(D) $4 + 12\sqrt{3}$

Grid-In

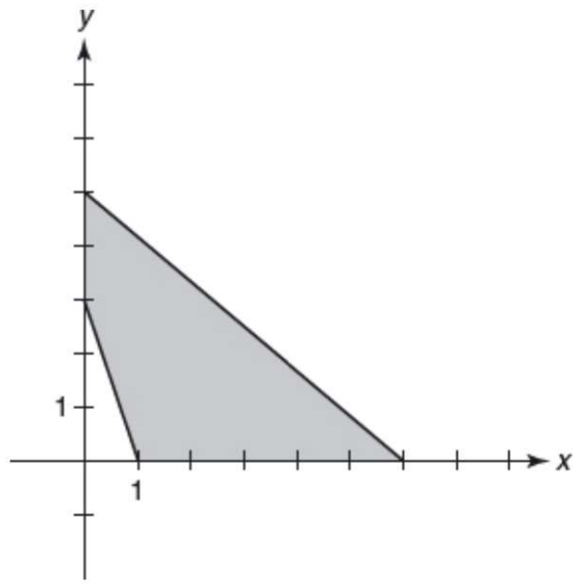
1. Brand *X* paint costs $14 per gallon, and 1 gallon provides coverage of an area of at most 150 square feet. What is the minimum cost of the amount of brand *X* paint needed to cover the four walls of a rectangular room that is 12 feet wide, 16 feet long, and 8 feet high?



2. What is the area of the square above?



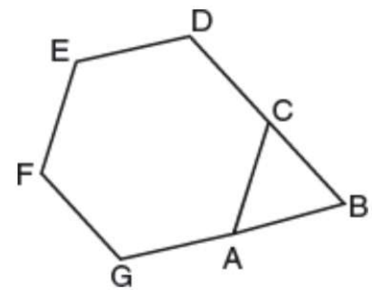
3. In the figure above, P and Q are the midpoints of sides AB and BC, respectively, of square $ABCD$. Line segment PB is extended by its own length to point E, and line segment PQ is extended to point F so that $FE \perp PE$. If the area of square $ABCD$ is 9, what is the area of quadrilateral $QBEF$?



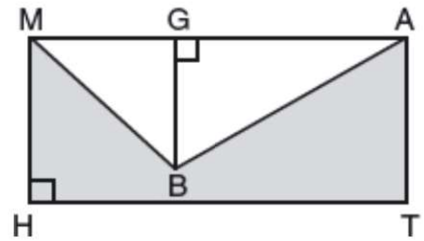
4. In the figure above, what is the number of square units in the area of the shaded region?

5. If the coordinates of the endpoints of a diagonal of a square are (−2, −3) and (5, 4), what is the number of square units in the area of the square?

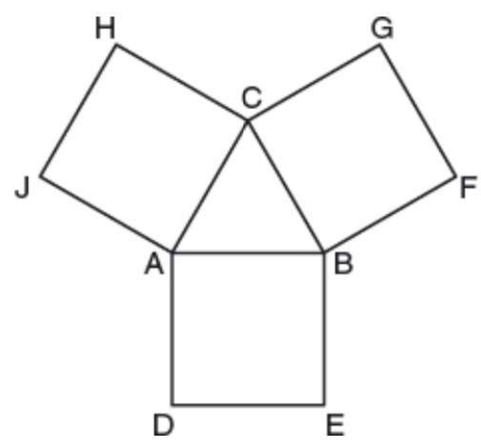
6. What is the number of square units in the area of the region in the first quadrant of the xy-plane that is bounded by $y = |x| + 2$, the line $x = 5$, the positive x-axis, and the positive y-axis?



7. In the figure above, $ACDEFG$ is a regular hexagon. Sides DC and GA are extended such that A is the midpoint of $\overline{BG}$ and C is the midpoint of $\overline{BD}$. If the area of $\triangle ABC$ is $9\sqrt{3}$ square centimeters, what is the number of centimeters in the perimeter of polygon $ABCDEFG$?



8. In the figure, $MATH$ is a rectangle, $GB = 4.8$, $MH = 6$, and $HT = 15$. The area of the shaded region is how many times larger than the area of $\triangle MBA$?



9. In the figure above, quadrilaterals $ABED$, $BFGC$, and $ACHJ$ are squares. If the area of equilateral $\triangle ABC$ is $16\sqrt{3}$ square inches, what is the number of inches in the perimeter of polygon $ADEBFGCHJA$?

LESSON 6-3 CIRCLES AND THEIR EQUATIONS

OVERVIEW

The SAT may also include questions that relate to basic relationships in a circle and its equation in the *xy*-coordinate plane.

CIRCLE TERMS

As illustrated in Figure 6.15,

- A **radius** of a circle is a segment whose endpoints are the center of the circle and a point on the circle. The plural of radius is radii.
- A **chord** of a circle is a segment whose endpoints are points on the circle. The longest chord of a circle is the **diameter,** which passes through the center of the circle. The radius length is one-half the length of the diameter of the circle.

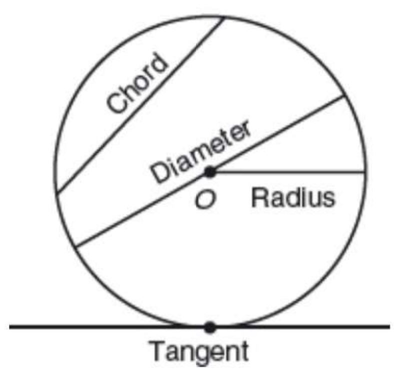


Figure 6.15 Segments related to a circle

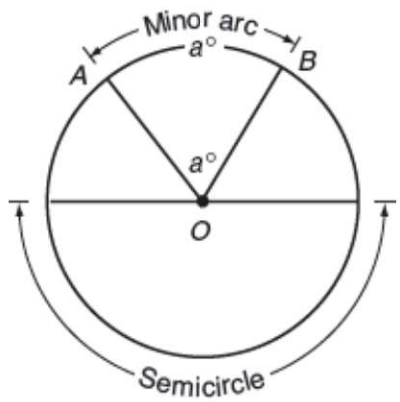


Figure 6.16 Angles and arcs of a circle

- A **tangent** of a circle is a line that intersects a circle in exactly one point.

ANGLES AND ARCS

Referring to Figure 6.16,

- An **arc** is a curved section of the circle. The number of degrees of arc in a circle is 360. A diameter divides a circle into two equal arcs, each of which is called a **semicircle**. An arc smaller than a semicircle is called a **minor arc** and is named by its endpoints. A **major arc** is an arc that is greater than a semicircle.
- An angle whose vertex is at the center and whose sides are radii is called a **central angle**. The measure of a central angle is equal to the degree measure of its intercepted arc and vice versa.

An **inscribed angle** is an angle whose vertex is on the circle and whose sides intercept an arc of a circle. Figure 6.17 illustrates that the measure of an inscribed angle is equal to one-half the measure of its intercepted arc. If $m\angle AC = x$, then the measure of inscribed angle ABC is $\frac{1}{2}x$.

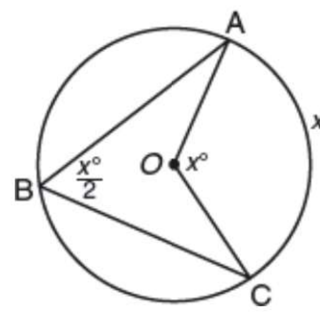
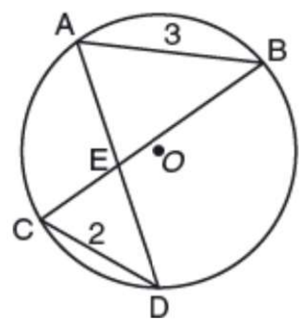


Figure 6.17 Comparing measures of central and inscribed angles.

> **TIP**
>
> **Inscribed angles that intercept the same arc of a circle are equal in measure.**



➡ Example

In circle O above, $AB = 3$ and $CD = 2$. Which of the following statements must be true?

I. $\angle ABC$ and $\angle DCB$ have equal measures
II. $\overline{AB} \parallel \overline{CD}$
III. $2(AE) = 3(CE)$

(A) I and II
(B) I and III
(C) I only
(D) III only

Solution

- Since inscribed angles BAD and DCB intercept the same arc, BD, they are equal in measure. Also, inscribed angles ABC and CDA intercept the same arc, AC, so their measures are also equal.

- Since triangles AEB and DEC agree in two pairs of angles, they are similar so the lengths of their corresponding sides have the same ratio. Since $\frac{AB}{CD} = \frac{3}{2}$, $\frac{AE}{CE} = \frac{3}{2}$ so $2(AE) = 3(CE)$, which makes statement III is true. There is no justification for concluding that $\overline{AB} \parallel \overline{CD}$ or that $\angle ABC$ and $\angle DCB$ have equal measures since they are *not* corresponding angles of similar triangles.

The correct choice is **(D)**.

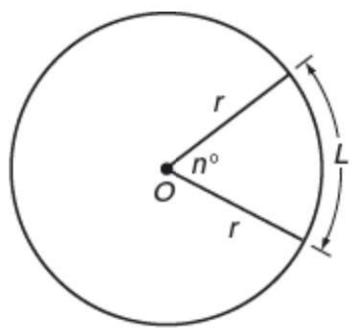
CIRCLE FORMULAS

The distance around a circle is its circumference. The circumference, C, of a circle depends on its diameter, d:

$$C = \pi d = 2\pi r$$

The length, L, of an arc of a circle is a fractional part of the circumference:

$$L = \frac{n}{360} \times \overbrace{2\pi r}^{\text{circumference}}$$

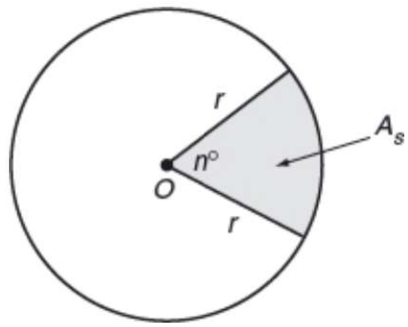


The area of a circle is the number of square units in the region it encloses. The area, A, of a circle depends on its radius:

$$A = \pi r^2$$

A sector of a circle is the region bounded by two radii and their intercepted arc. The area, A_S, of a sector of a circle is a fractional part of the area of the circle:

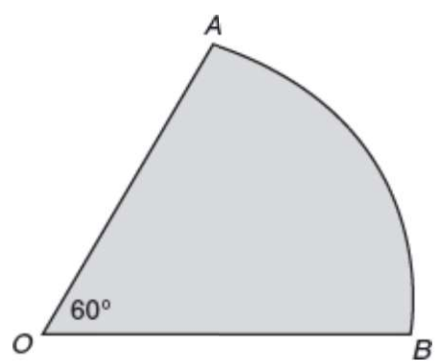
$$A_s = \frac{n}{360} \times \overbrace{\pi r^2}^{\text{area of circle}}$$



➡ Example

If in the accompanying figure the length of arc AB of circle O is 8π, what is the number of square units in the area of the shaded region?

(A) 16π
(B) 32π
(C) 64π
(D) 96π



Solution

- Find the radius of circle O:

$$L = \frac{60}{360} \times 2\pi r = 8\pi$$

$$\frac{120}{360} \times r = 8$$

$$\frac{1}{3} r = 8$$

$$r = 8 \times 3 = 24$$

- Find the area of the shaded region:

$$A_s = \frac{60}{360} \times \pi(24)^2$$
$$= \frac{\pi}{6} \times 576$$
$$= 96\pi$$

The correct choice is **(D)**.

TANGENTS AND RADII

A radius drawn to the point of contact of a tangent is perpendicular to the tangent at that point, as shown in Figure 6.18.

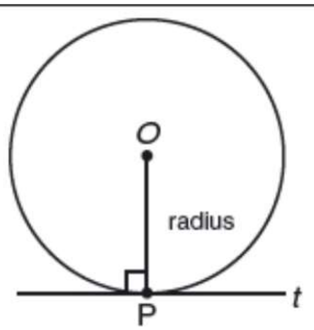


Figure 6.18 Radius $OP \perp$ tangent t

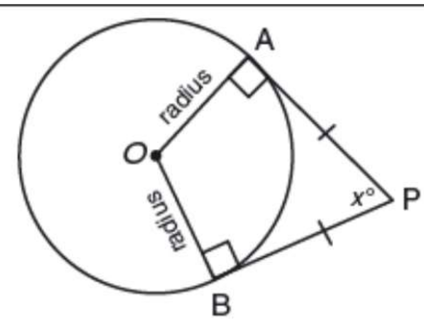


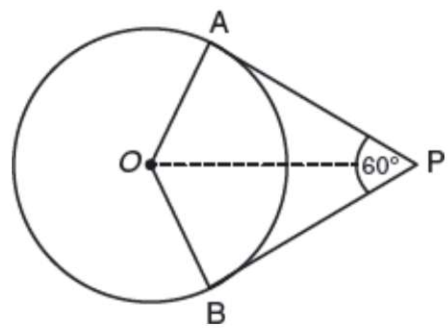
Figure 6.19 Tangents drawn to a circle from the same exterior point

If two tangents are drawn to a circle from the same exterior point P, as shown in Figure 6.19, then the following relationships are always true:

- The tangents have the same length so $PA = PB$.

- Opposite angles of the quadrilateral that is formed are supplementary. If the measure of $\angle P$ is x, then the measure of central angle AOB is $180 - x$.
- The segment drawn from point P to the center of the circle (not shown) divides the quadrilateral into two congruent right triangles and, as a result, bisects angles AOB and APB, as well as arc AB.

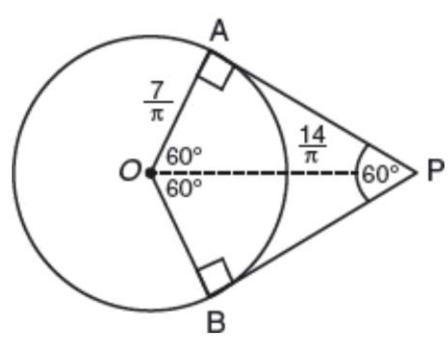
➡ Example



In the figure above, line segments PA and PB are tangent to circle O at points A and B, respectively, and the measure of $\angle APB$ is 60°. Segment OP is drawn. If $OP = \frac{14}{\pi}$, what is the length of minor arc AB?

Solution

- Angle AOB measures $180° - 60° = 120°$. Since OP divides quadrilateral $OAPB$ into two congruent right triangles, it bisects $\angle AOB$:



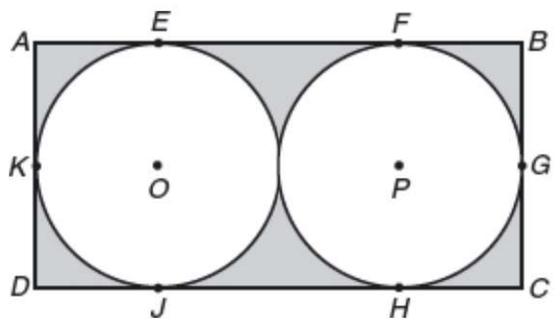
- The side opposite the 30° angle in a right triangle is one-half the length of the hypotenuse so $OA = \frac{7}{\pi}$.
- Since the measure of minor arc AB is 120°:

$$\text{Length of } AB = \frac{120}{360} \times 2\pi \times \frac{7}{\pi}$$
$$= \frac{1}{3} \times 14$$
$$= \frac{14}{3}$$

Grid-in **14/3**

FINDING AREAS OF SHADED REGIONS INDIRECTLY

To find the area of a shaded region, you may need to subtract the areas of figures that overlap.

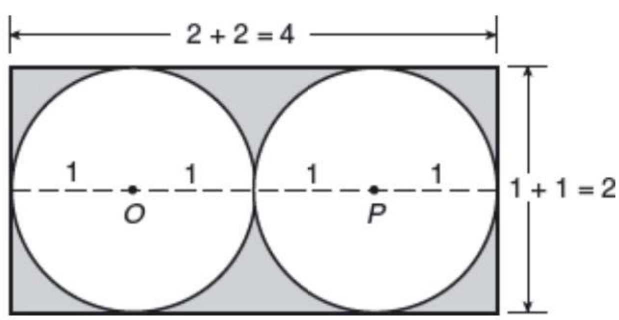


➥ Example

In the figure above, the radius of each circle is 1. If circles *O* and *P* touch the sides of rectangle *ABCD* only at the lettered points, what is the area of the shaded region?

Solution

The area of the shaded regoin is equal to the area of the rectangle minus the sum of the areas of the two circles.



- The width of the rectangle is equal to the diameter of a circle, which is $1 + 1$ or 2.
- The length of the rectangle is equal to the sum of the two diameters, which is $2 + 2$ or 4.
- The area of the rectangle is length × width $= 4 \times 2 = 8$, and the area of the each circle is $\pi r^2 = \pi 1^2 = \pi$.
- Thus,

$$\text{Area of shaded region} = 8 - (\pi + \pi) = \mathbf{8 - 2\pi}$$

DIAMETER AND CHORD RELATIONSHIPS

In a circle,

- If a line is drawn through the center of a circle and perpendicular to a chord, it bisects the chord and its arcs. See Figure 6.20.

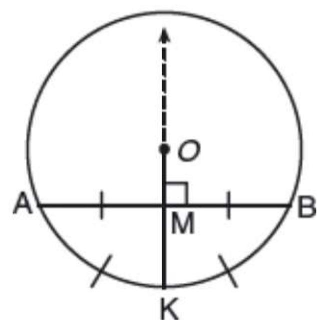


Figure 6.20 If $\overline{OK} \perp \overline{AB}$, then $AM = BM$ and $m\angle AK = m\angle BK$

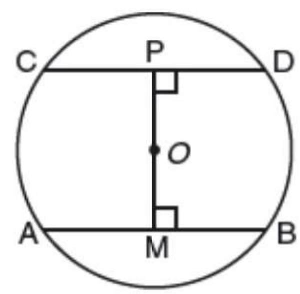
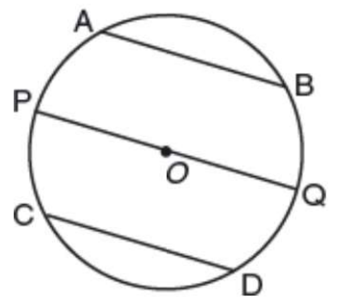


Figure 6.21 If $\overline{AB} \cong \overline{CD}$, then $OM = OP$; If $OM = OP$, then $\overline{AB} \cong \overline{CD}$

- Congruent chords are the same distance from the center of the circle, and chords that are the same distance from the center of a circle are congruent. See Figure 6.21.

➡ Example



In the figure above, $\overline{PQ}$ is a diameter of circle O, $\overline{AB} \parallel \overline{PQ} \parallel \overline{CD}$, and $\overline{AB} \cong \overline{CD}$. The length of chord $\overline{AB}$ is $\frac{3}{4}$ of the length of diameter $\overline{PQ}$. If r represents the radius length of circle O, what is the distance between chords $\overline{AB}$ and $\overline{CD}$ in terms of r?

(A) $\frac{5}{4}r$

(B) $\frac{\sqrt{7}}{4}r$

(C) $\frac{3}{2}r$

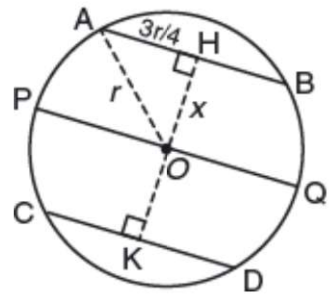
(D) $\frac{\sqrt{7}}{2}r$

> **TIP**
>
> **When working in a circle in which the radius is not drawn, it may be necessary to draw your own radius in a way that helps solve the problem.**

Solution

- It is given that $AB = \frac{3}{4} \times 2r = \frac{3}{2}r$. Draw $\overline{OH} \perp \overline{AB}$ and extend $\overline{OH}$ so it intersects $\overline{CD}$ at K. Since $\overline{AB} \parallel \overline{CD}$, $\overline{OK} \perp \overline{CD}$, HK represents the distance between the two chords.

- Draw radius OA thereby forming right triangle OHA. Because $OH \perp AB$, it bisects AB so $AH = \frac{1}{2} \times \frac{3}{2}r = \frac{3}{4}r$:



- Use the Pythagorean theorem to find the length of $\overline{OH}$:

$$x^2 + \left(\frac{3}{4}r\right)^2 = r^2$$

$$x^2 + \frac{9}{16}r^2 = r^2$$

$$x^2 = \frac{7}{16}r^2$$

$$\sqrt{x^2} = \sqrt{\frac{7}{16}r^2}$$

$$x = \frac{\sqrt{7}}{4}r$$

- Since $\overline{AB} \cong \overline{CD}$, $OH = OK = \frac{\sqrt{7}}{4}r$ and $HK = 2 \times \frac{\sqrt{7}}{4}r = \frac{\sqrt{7}}{2}r$.

The correct choice is **(D)**.

EQUATION OF A CIRCLE: CENTER-RADIUS FORM

The set of points (x, y) in the xy-coordinate plane that lie on a circle whose center is at (h, k) and has a radius of r, as shown in Figure 6.22, is described by the equation $(x - h)^2 + (y - k)^2 = r^2$.

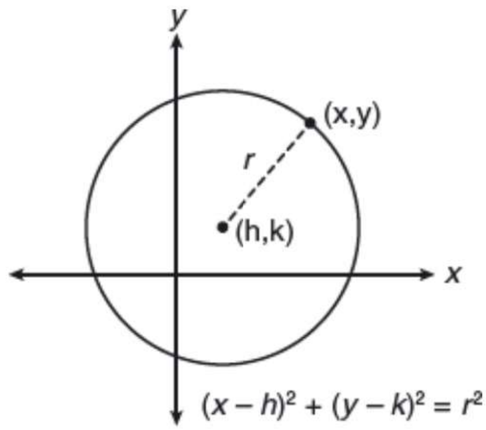


Figure 6.22 Equation of a circle in the *xy*-plane

The equation $(x - h)^2 + (y - k)^2 = r^2$ is referred to as the **center-radius form** of the equation of a circle. If the circle is centered at the origin, then $h = k = 0$ and the equation of the circle simplifies to $x^2 + y^2 = r^2$.

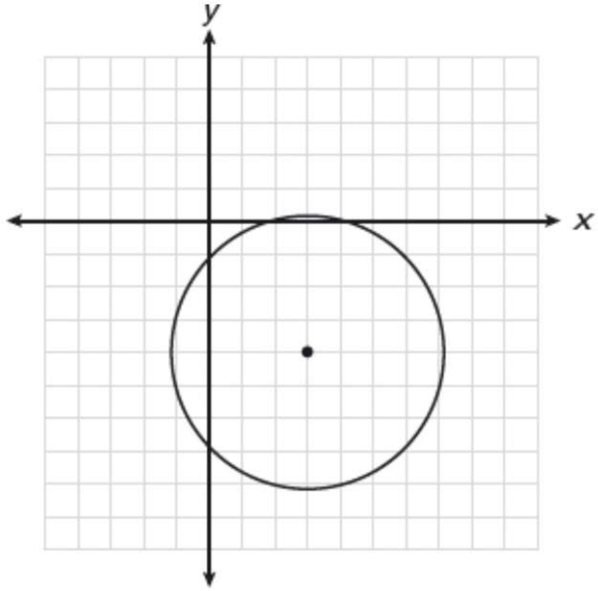
- If the center of a circle is at (−2, 1) and its radius is 5, then an equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$ where $h = -2$, $k = 1$, and $r = 5$. Making the substitutions gives $(x + 2)^2 + (y - 1)^2 = 25$.
- The center and radius of a circle can be read directly from the center-radius form of its equation. If the equation of a circle is given as

$$(x + 1)^2 + (y - 5)^2 = 64$$

then you know that $h = -1$, $k = 5$, and $r^2 = 64$ so $r = 8$. Hence, the center of the circle is (−1, 5), and the radius is 8.

➡ Example

Which equation could represent the circle shown in the graph below that passes through the point (0, −1)?



(A) $(x - 3)^2 + (y + 4)^2 = 16$
(B) $(x + 3)^2 + (y - 4)^2 = 18$
(C) $(x + 3)^2 + (y - 4)^2 = 16$
(D) $(x - 3)^2 + (y + 4)^2 = 18$

Solution

Counting unit boxes, the center of the circle is located at (3, −4) so the equation of the circle has the form $(x - 3)^2 + (y + 4)^2 = r^2$. Therefore, you can eliminate choices (B) and (C). Since the radius is more than 4 units, $r^2 > 16$, which means the correct choice must be **(D)**.

➡ Example

What is the center and radius of a circle whose equation is $3x^2 + 3y^2 - 12x + 18y = 69$?

Solution

Rewrite the equation in center-radius form by completing the square for both x and y.

- Divide each term of the equation by 3, and then collect terms with same variable:

$$(x^2 - 4x) + (y^2 + 5y) = 23$$

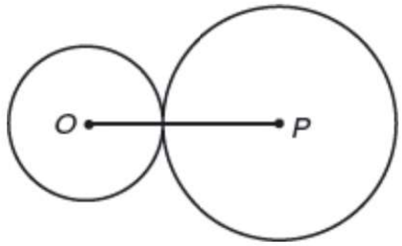
- Complete the square for x and for y:

$$(x^2 - 4x + \underline{\underline{4}}) + (y^2 + 6y + \underline{\underline{9}}) = 23 + \underline{\underline{4 + 9}}$$
$$(x - 2)^2 + (y + 3)^2 = 36$$

- The circle equation now has the center-radius form $(x - h)^2 + (y - k)^2 = r^2$ with center at $(h, k) = (2, -3)$ and radius $r = \sqrt{36} = 6$.

LESSON 6-3 TUNE-UP EXERCISES

Multiple-Choice



1. Circles O and P intersect at exactly one point, as shown in the figure above. If the radius of circle O is 2 and the radius of circle P is 6, what is the circumference of any circle that has OP as a diameter?

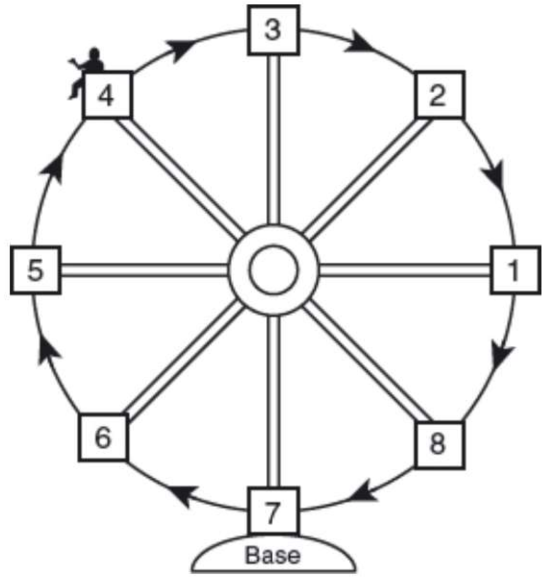
   (A) 4π
   (B) 8π
   (C) 12π
   (D) 16π

2. What is the area of a circle with a circumference of 10π?

   (A) $\sqrt{10\pi}$
   (B) 5π
   (C) 25π
   (D) 100π

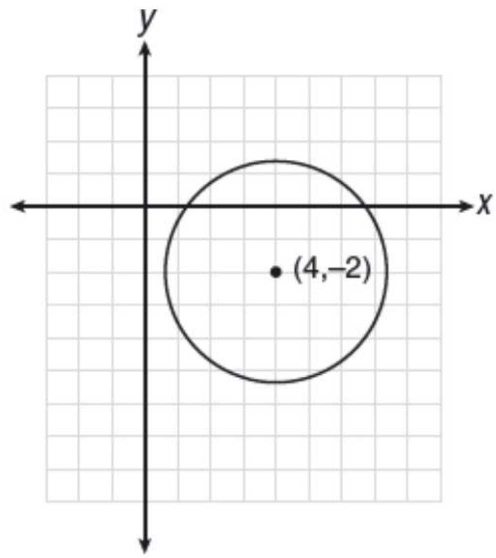
3. Every time the pedals go through a 360° rotation on a certain bicycle, the tires rotate three times. If the tires are 24 inches in diameter, what is the minimum number of complete rotations of the pedals needed for the bicycle to travel at least 1 mile? [1 mile = 5,280 feet]

   (A) 24
   (B) 281
   (C) 561
   (D) 5,280



4. Kristine is riding in car 4 of the ferris wheel represented in the diagram above, which is $\frac{84}{\pi}$ meters from car 8. The ferris wheel is rotating in the direction indicated by the arrows. If each of the cars are equally spaced around the circular wheel, what is the best estimate of the number of meters in the distance through which Kristine's car will travel to reach the bottom of the ferris wheel before her car returns to the same position?

(A) 42.0
(B) 52.50
(C) 64.75
(D) 105.0

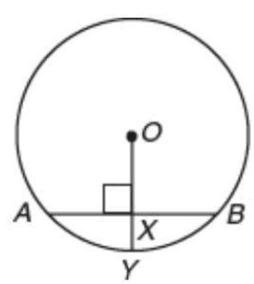


5. Which of the following could be an equation of the circle above?

(A) $(x-4)^2+(y+2)^2=17$
(B) $(x+4)^2+(y-2)^2=17$
(C) $(x-4)^2+(y+2)^2=13$
(D) $(x+4)^2+(y-2)^2=13$

6. If the equation of a circle is $x^2-10x+y^2+6y=-9$, which of the following lines contains a diameter of the circle?

(A) $y=2x-7$
(B) $y=-2x+7$
(C) $y=2x+13$
(D) $y=-2x+13$



7. In the figure above, if the radius length of circle O is 10, $\overline{OY}\perp\overline{AB}$, and $AB=16$, what is the length of segment XY?

(A) 2

(B) 3
(C) 4
(D) 6

8. If a bicycle wheel has traveled $\frac{f}{\pi}$ feet after n complete revolutions, what is the length in feet of the diameter of the bicycle wheel?

(A) $\frac{f}{n\pi^2}$

(B) $\frac{\pi^2}{fn}$

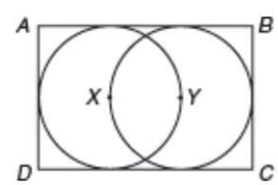
(C) $\frac{nf}{\pi^2}$

(D) nf

$$x^2 + y^2 - 6x + 8y = 56$$

9. What is the area of a circle whose equation is given above?

(A) 25π
(B) 81π
(C) 162π
(D) $6{,}561\pi$



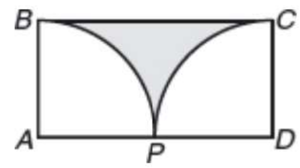
10. In the figure above, X and Y are the centers of two overlapping circles. If the area of each circle is 7, what is the area of rectangle $ABCD$?

(A) $14 - \frac{17}{\pi}$

(B) $7 + \frac{14}{\pi}$

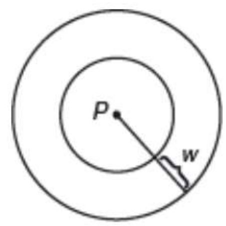
(C) $\frac{28}{\pi}$

(D) $\frac{42}{\pi}$



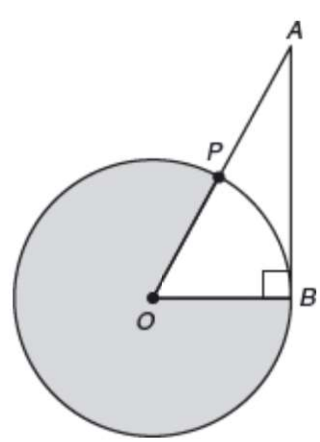
11. In rectangle *ABCD* above, arcs *BP* and *CP* are quarter circles with centers at points *A* and *D*, respectively. If the area of each quarter circle is π, what is the area of the shaded region?

(A) $4-\frac{\pi}{2}$
(B) $4-\pi$
(C) $8-\pi$
(D) $8-2\pi$



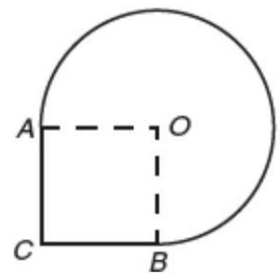
12. In the figure above, point *P* is the center of each circle. The circumference of the larger circle exceeds the circumference of the smaller circle by 12π. What is the width, *w*, of the region between the two circles?

(A) 4
(B) 6
(C) 8
(D) 9



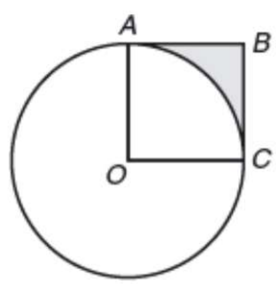
13. The circle shown above has center O and a radius length of 12. If P is the midpoint of $\overline{OA}$ and $\overline{AB}$ is tangent to circle O at B, what is the area of the shaded region?

(A) 81π
(B) 96π
(C) 120π
(D) 128π



14. In the figure above, $OACB$ is a square with area $4x^2$. If OA and OB are radii of a sector of a circle O, what is the perimeter, in terms of x, of the unbroken figure?

(A) $x(4 + 3\pi)$
(B) $x(3 + 4\pi)$
(C) $x(6 + 4\pi)$
(D) $4(x + 2\pi)$



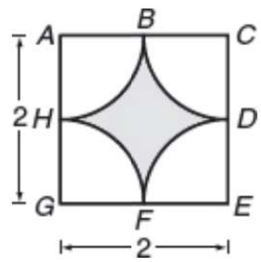
15. In the figure above, $OABC$ is a square. If the area of circle O is 2π, what is the area of the shaded region?

(A) $\frac{\pi}{2} - 1$

(B) $2 - \frac{\pi}{2}$

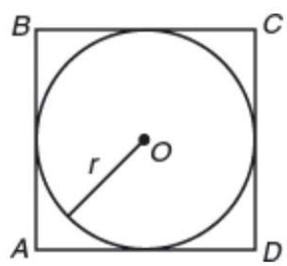
(C) $\pi - 2$

(D) $\frac{\pi - 1}{2}$



16. In the figure above, the vertices of square *ACEG* are the centers of four quarter circles of equal area. What is the best approximation for the area of the shaded region? (Use $\pi = 3.14$.)

(A) 0.64
(B) 0.79
(C) 0.86
(D) 1.57



17. In the figure above, if circle *O* is inscribed in square *ABCD* in such a way that each side of the square is tangent to the circle, which of the following statements must be true?

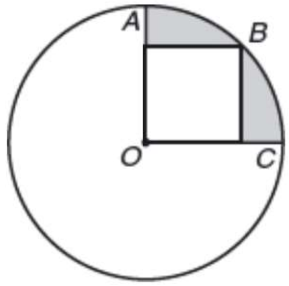
I. $AB \times CD < \pi \times r \times r$

II. Area $ABCD = 4r^2$

III. $r < \frac{2(CD)}{\pi}$

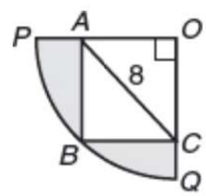
(A) I and II
(B) I and III

(C) II and III
(D) II only



18. In the figure above, $OABC$ is a square and B is a point on the circle with center O. If $AB = 6$, what is the area of the shaded region?

(A) $9(\pi - 2)$
(B) $9(\pi - 1)$
(C) $12(\pi - 2)$
(D) $18(\pi - 2)$

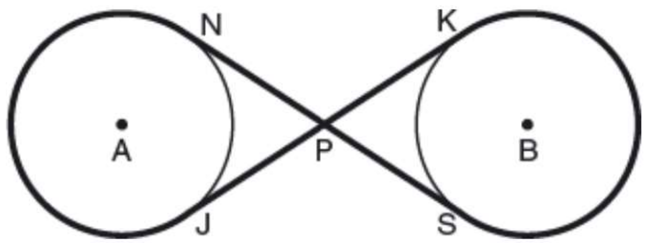


19. In the figure above, arc PBQ is one-quarter of a circle with center at O, and $OABC$ is a rectangle. If AOC is an isosceles right triangle with $AC = 8$, what is the perimeter of the figure that encloses the shaded region?

(A) $24 - 4\pi$
(B) $24 - 4\sqrt{2} + 4\pi$
(C) $16 - 4\sqrt{2} + 4\pi$
(D) $16 + 4\pi$

20. The center of circle Q has coordinates $(3, -2)$ in the xy-plane. If an endpoint of a radius of circle Q has coordinates $R(7, 1)$, what is an equation of circle Q?

(A) $(x - 3)^2 + (y + 2)^2 = 5$
(B) $(x + 3)^2 + (y - 2)^2 = 25$
(C) $(x - 3)^2 + (y + 2)^2 = 25$
(D) $(x + 3)^2 + (y - 2)^2 = 5$



21. In the pully system illustrated in the figure above, a belt with negligible thickness is stretched tightly around two identical wheels represented by circles *A* and *B*. If the radius of each wheel is $\frac{12}{\pi}$ inches and the measure of $\angle NPJ$ is 60°, what is the length of the belt?

(A) $32+\frac{48\sqrt{3}}{\pi}$

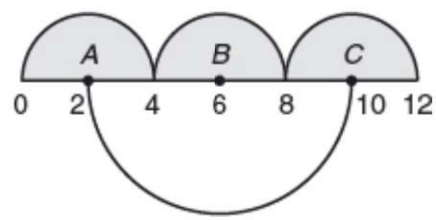
(B) $48+\frac{32\sqrt{3}}{\pi}$

(C) $(32+48\sqrt{3})\pi$

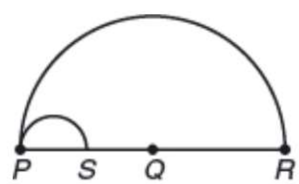
(D) $80\pi\sqrt{3}$

Grid-In

1.



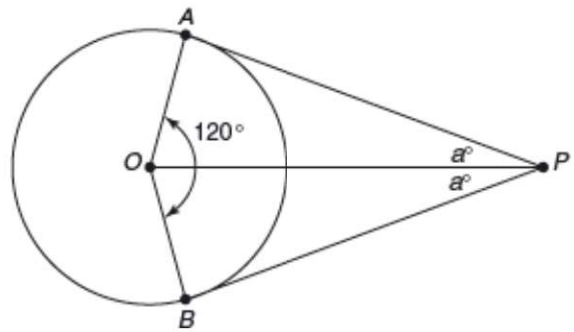
In the figure above, the sum of the areas of the three shaded semicircles with centers at *A*, *B*, and *C* is *X*, and the area of the larger semicircle below the line is *Y*. If $Y-X=k\pi$, what is the value of *k*?



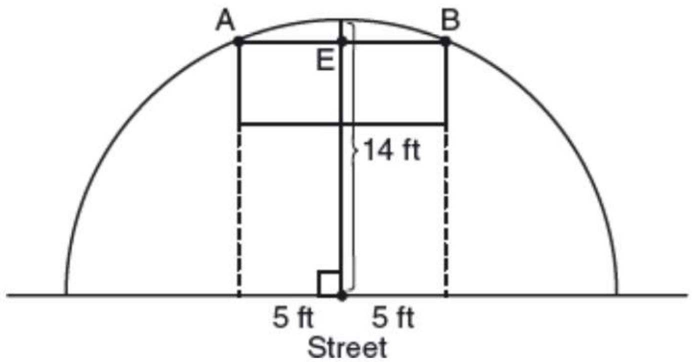
2. In the figure above, each arc is a semicircle. If *S* is the midpoint of *PQ* and *Q* is the midpoint of *PR*, what is the ratio of the area of semicircle

PS to the area of semicircle *PR*?

3. What is the distance in the *xy*-plane from the point (3, −6) to the center of the circle whose equation is $x(x + 4) + y(y - 12) = 9$?



4. In the figure above, $\overline{PA}$ is tangent to circle *O* at point *A*, $\overline{PB}$ is tangent to circle *O* at point *B*. Angle *AOB* measures 120° and $\overline{OP} = \frac{24}{\pi}$. What is the length of minor arc *AB*?



5. The diagram above shows a semicircular arch over a street that has a radius of 14 feet. A banner is attached to the arch at points *A* and *B*, such that *AE* = *EB* = 5 feet. How many feet above the ground are these points of attachment for the banner, correct to the *nearest tenth* of a foot?

LESSON 6-4 SOLID FIGURES

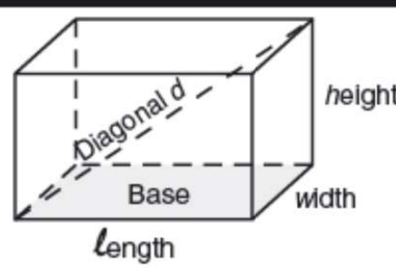
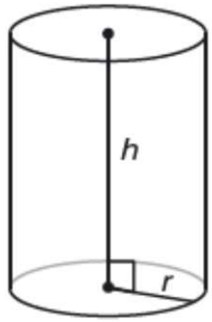
OVERVIEW

The reference section at the beginning of each SAT Math section includes the volume formulas for these solids: rectangular box, right circular cylinder, sphere, right circular cone, and pyramid with a rectangular base.

VOLUME OF A RECTANGULAR SOLID AND CYLINDER

The SAT may include problems that make use of the formulas in Table 6.1. The **lateral area** of a right circular cylinder is the area of its curved surface that does not include the bases. The **surface area** (SA) of a solid is the sum of the areas of all of its surfaces.

Table 6.1 Formulas for a Rectangular Box and Circular Cylinder

Type of Solid	Diagram	Formulas
Rectangular box		• $V = l \times w \times h$ • $d = \sqrt{l^2 + w^2 + h^2}$ • $SA = 2[l \times w + l \times h + w \times h]$
Right circular cylinder		• $V = \pi r^2 h$ • $SA = \underbrace{2\pi r^2}_{\text{Area of bases}} + \underbrace{2\pi rh}_{\text{Lateral area}}$

➡ Example

An oil filter that has the shape of a right circular cylinder is 12 centimeters in height and has a volume of 108π cubic centimeters. What is the *diameter* of the base of the cylinder, in centimeters?

Solution

$$V = \pi r^2 h$$

$$\frac{108\pi}{12\pi} = \frac{12\pi(r^2)}{12\pi}$$

$$9 = r^2$$

$$3 = r$$

Since the radius of the base is 3 centimeters, the diameter is 6 centimeters.

➡ Example

A coffee shop makes coffee in a 5 gallon capacity urn and serves it in cylindrical-shaped mugs with an internal diameter of 3 inches. Coffee is poured into each mug at a height of about 4 inches. What is the largest number of full mugs of coffee that can be filled from the coffee urn if the urn is filled to capacity? (Note: There are 231 cubic inches in 1 gallon.)

Solution

- Five gallons is equivalent in volume to 231 × 5 = 1,155 cubic inches.
- Find the volume of each filled coffee mug:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi\left(\frac{3}{2}\right)^2 (4) \\ &= \cancel{4}\pi\left(\frac{9}{\cancel{4}}\right) \\ &= 9\pi \text{ cubic inches} \end{aligned}$$

- Find the number of filled coffee mugs by dividing the capacity of the urn by 9π:

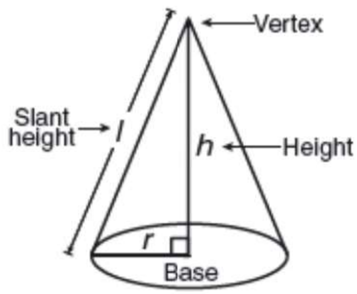
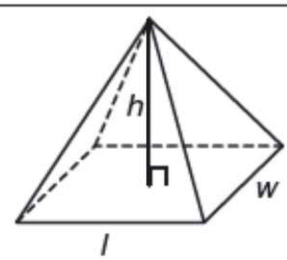
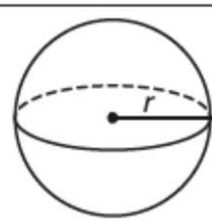
$$\frac{1{,}155}{9\pi} \approx 40.85$$

Hence, **40** full coffee mugs can be filled from the coffee urn.

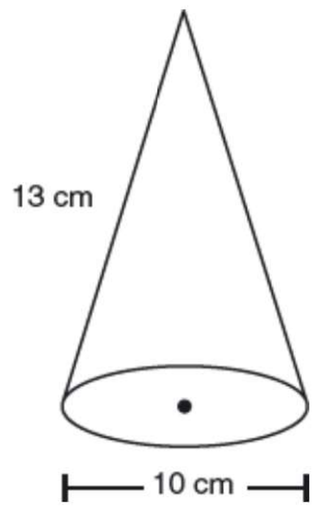
VOLUME OF A CONE, PYRAMID, AND SPHERE

The volume of a right circular cone and a pyramid are each calculated by multiplying one-third of the area of the base by the height of the figure. The volume of a sphere depends on its radius. See Table 6.2 where B represents the area of the base of the solid.

Table 6.2 Volume Formulas for a Cone, Pyramid, and Sphere

Type of Solid	Diagram	Formulas
Right circular cone		$V = \frac{1}{3}Bh$ $= \frac{1}{3}\pi r^2 h$
Pyramid with rectangle base		$V = \frac{1}{3}Bh$ $= \frac{1}{3}l \times w \times h$
Sphere		$V = \frac{4}{3}\pi r^3$

➥ Example

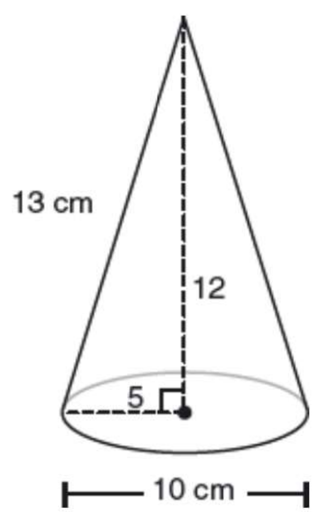


What is the volume, in cubic centimeters, of the right circular cone in the figure above, in terms of π?

Solution

Use the volume formula $V = \frac{1}{3}\pi r^2 h$ where $r = \frac{1}{2}(10) = 5$, and h = 12 since it is a leg in a 5-***12***-13 right triangle:

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi \times 5^2 \times 12$$
$$= 100\pi \text{ cubic centimeters}$$



LESSON 6-4 TUNE-UP EXERCISES

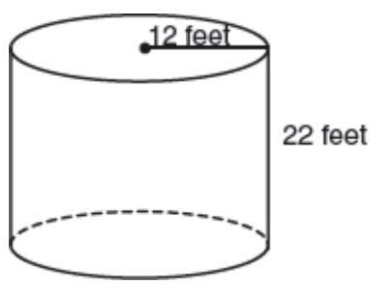
Multiple-Choice

1. A pyramid has a height of 12 centimeters and a square base. If the volume of the pyramid is 256 cubic centimeters, how many centimeters are in the length of one side of its base?

   (A) 8
   (B) 16
   (C) 32
   (D) $\frac{8}{\sqrt{3}}$

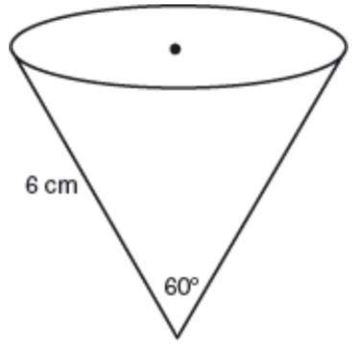
2. The volume of a rectangular box is 144 cubic inches. The height of the solid is 8 inches. Which measurements, in inches, could be the dimensions of the base?

   (A) 3.3 × 5.5
   (B) 2.5 × 7.2
   (C) 12.0 × 8.0
   (D) 9.0 × 9.0



3. The cylindrical tank shown in the diagram above is to be painted. The tank is open at the top, and the bottom does *not* need to be painted. Only the outside needs to be painted. Each can of paint covers 500 square feet. What is the least number of cans of paint that must be purchased to complete the job?

   (A) 2
   (B) 3
   (C) 4
   (D) 5



4. In the figure above of a right circular cone, what is the approximate number of cubic centimeters in the volume of the cone?

   (A) 49
   (B) 105
   (C) 210
   (D) 306

5. Sophie has a hard rubber ball whose circumference measures 13 inches. She wants to store it in a box. What is the number of cubic inches in the volume of the smallest cube-shaped box with integer dimensions that she can use?

   (A) 64
   (B) 81
   (C) 125
   (D) 216

6. The amount of light produced by a cylindrical fluorescent light bulb depends on its lateral area. A certain cylindrical fluorescent light bulb is 36 inches long, has a 1 inch diameter, and is manufactured to produce 0.283 watts of light per square inch. What is the best estimate for the total amount of light that it is able to produce?

   (A) 32 watts
   (B) 34 watts
   (C) 40 watts
   (D) 48 watts

7. A rectangular fish tank has a base 2 feet wide and 3 feet long. When the tank is partially filled with water, a solid cube with an edge length of 1 foot is placed in the tank. If no overflow of water from the tank is

assumed, by how many *inches* will the level of the water in the tank rise when the cube becomes completely submerged?

(A) $\frac{1}{6}$
(B) $\frac{1}{2}$
(C) 2
(D) 3

8. The volume of a cylinder of radius r is $\frac{1}{4}$ of the volume of a rectangular box with a square base of side length x. If the cylinder and the box have equal heights, what is r in terms of x?

(A) $\frac{x^2}{2\pi}$

(B) $\frac{x}{2\sqrt{\pi}}$

(C) $\frac{\sqrt{2x}}{\pi}$

(D) $\frac{\pi}{\sqrt{2x}}$

9. The height of sand in a cylinder-shaped can drops 3 inches when 1 cubic foot of sand is poured out. What is the diameter, in *inches*, of the cylinder?

(A) $\frac{2}{\sqrt{\pi}}$

(B) $\frac{4}{\sqrt{\pi}}$

(C) $\frac{16}{\pi}$

(D) $\frac{48}{\sqrt{\pi}}$

10. The height h of a cylinder equals the circumference of the cylinder. In terms of h, what is the volume of the cylinder?

(A) $\frac{h^3}{4\pi}$

(B) $\frac{h^2}{2\pi}$

(C) $\frac{h^3}{2}$

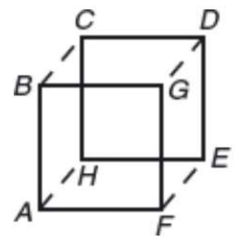
(D) $h^2 + 4\pi$

11. As shown in the figure above, a worker uses a cylindrical roller to help pave a road. The roller has a radius of 9 inches and a width of 42 inches. To the *nearest square inch*, what is the area the roller covers in one complete rotation?

(A) 2,374
(B) 2,375
(C) 10,682
(D) 10,688

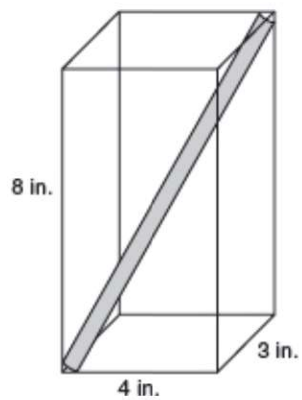
12. The density of lead is approximately 0.41 pounds per cubic inch. What is the approximate mass, in pounds, of a lead ball that has a 5 inch diameter?

(A) 26.8
(B) 78.5
(C) 80.4
(D) 214.7



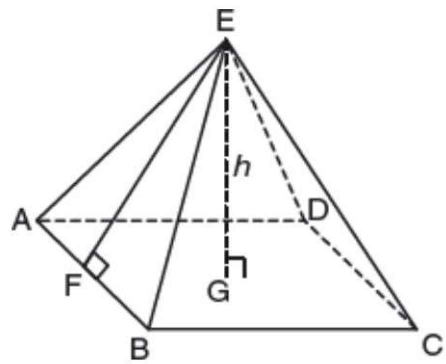
13. In the figure above, if the edge length of the cube is 4, what is the shortest distance from A to D?

(A) $4\sqrt{2}$
(B) $4\sqrt{3}$
(C) 8
(D) $4\sqrt{2} + 4$



14. A cylindrical tube with negligible thickness is placed into a rectangular box that is 3 inches by 4 inches by 8 inches, as shown in the accompanying diagram. If the tube fits exactly into the box diagonally from the bottom left corner to the top right back corner, what is the best approximation of the number of inches in the length of the tube?

(A) 3.9
(B) 5.5
(C) 7.8
(D) 9.4



15. In the pyramid shown in the diagram above G is the center of square base $ABCD$, $\overline{EF} \perp \overline{AB}$, and h the height of the pyramid. Which statements must be true?

I. $EA = EC$
II. $\triangle BEC$ is isosceles
III. $EF = EG$

(A) I and II only
(B) I and III only
(C) I only
(D) II only

16. If a pyramid with a square base with side length s and a right cone with radius r have equal heights and equal volumes, then which equation must be true?

(A) $s = \sqrt{\pi r}$

(B) $s = \frac{\sqrt{r}}{\pi}$

(C) $s = \pi\sqrt{r}$

(D) $s = r\sqrt{\pi}$

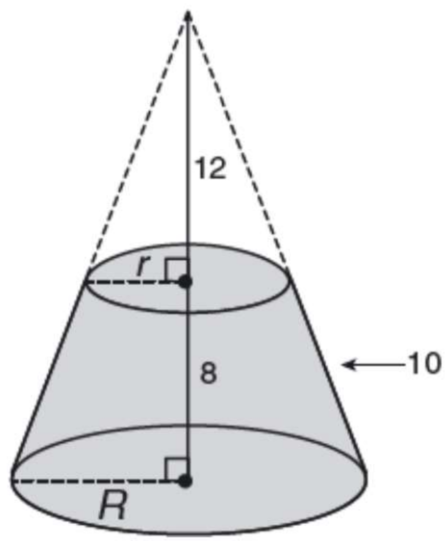
17. An ice cube has a surface area of 150 square centimeters. If the ice cube melts at a constant rate of 13.0 cubic centimeters per minute, the number of minutes that elapse before the ice cube is completely melted is closest to which of the following?

(A) 10
(B) 11
(C) 12
(D) 14

18. A hot water tank with a capacity of 85.0 gallons of water is being designed to have the shape of a right circular cylinder with a diameter 1.8 feet. Assuming that there are 7.48 gallons in 1 cubic foot, how high in feet will the tank need to be?

(A) 4.50
(B) 4.75
(C) 5.00
(D) 5.25

For **Questions 19–20** use the figure below.



A lamp shade with a circular base is an example of a solid shape called a *frustrum*. In the figure above, the shaded region represents a frustrum of a right cone in which the portion of the original cone that lies 12 inches below its vertex has been cut off by a slicing plane (not shown) parallel to the base.

19. If the height and slant height of the frustrum are 8 inches and 10 inches, respectively, what is the number of inches in the radius length, R, of the original cone?

(A) 9
(B) 12
(C) 15
(D) 18

20. What is the volume, in cubic inches, of the frustrum?

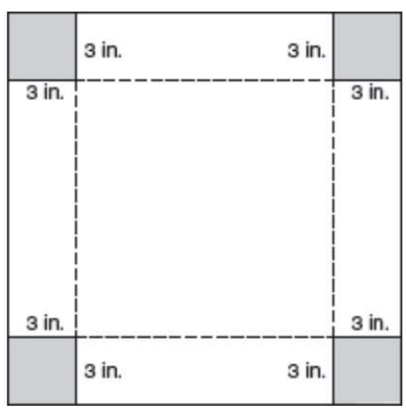
(A) 324π
(B) 812π
(C) $1{,}089\pi$
(D) $1{,}176\pi$

Grid-In

1. The dimensions of a rectangular box are integers greater than 1. If the area of one side of this box is 12 and the area of another side is 15, what is the volume of the box?

2. The Parkside Packing Company needs a rectangular shipping box. The box must have a length of 1 foot, a width of 8 inches, and a volume of *at least* 700 cubic inches. What is the least number of inches in height of the box such that the height is a whole number?

3. A planned building was going to be 100 feet long, 75 feet deep, and 30 feet high. The owner decides to increase the volume of the building by 10% without changing the dimensions of the depth and the height. What will be the new length of this building in feet?



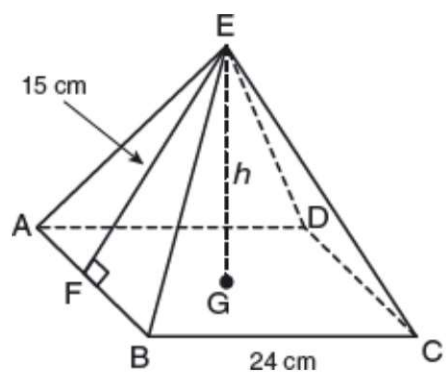
Note: Figure is not drawn to scale.

4. A box is constructed by cutting 3-inch squares from the corners of a square sheet of cardboard, as shown in the accompanying diagram, and then folding the sides up. If the volume of the box is 75 cubic inches,

find the number of square inches in the area of the *original* sheet of cardboard.

5. Two spheres that are tangent to each other have volumes of 36π cubic centimeters and 972π cubic centimeters. What is the greatest possible distance, in centimeters, between a point on one sphere and a second point on the other sphere?

6. A sealed cylindrical can holds three tennis balls each with a diameter of 2.5 inches. If the can is designed to have the smallest possible volume, find the number of cubic inches of unoccupied space inside the can correct to the *nearest tenth of a cubic inch*.



7. The *slant height* of a pyramid is the perpendicular distance from the vertex of the pyramid to the base of a triangular side. In the pyramid shown in the figure above, the height is labeled h, the length of a side of the square base is 24 cm, and the slant height is 15 cm. What is the volume, in cubic centimeters, of the pyramid?

8. A bookend that weights 0.24 pounds is shaped like a pyramid with a square base. How many pounds does a larger, similar pyramid-shaped bookend weigh if it is made of the same material and each corresponding dimension is $2\frac{1}{2}$ times as large?

LESSON 6-5 BASIC TRIGONOMETRY

OVERVIEW

This section reviews some basic trigonometric facts and relationships that you need to know for the new SAT. The Greek letter θ (theta) is sometimes used to represent the unknown measure of an angle.

MEASURING ANGLES IN RADIANS

Degrees are a measure of rotation where $1° = \frac{1}{360}$th of a complete rotation. A *radian* is a unit of angle measure that express the measure of an angle as a real number. One **radian** is the measure of a central angle of a circle that intercepts an arc whose length is the same as a radius of that circle. In Figure 6.23, central angle θ intercepts an arc that is 6π inches in length in a circle whose radius measures 12 inches. The measure of angle θ, in radians, is $\frac{6\pi}{12} = \frac{\pi}{2}$ radians.

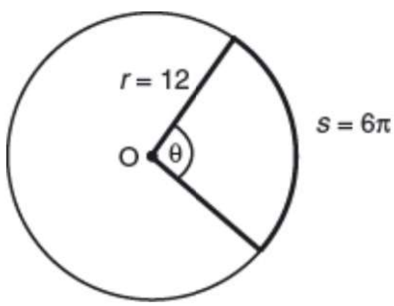


Figure 6.23 Radian measure: $\theta = \frac{s}{r}$

In general, $\theta = \frac{s}{r}$ or $s = r \cdot \theta$ where s represents the length of the arc intercepted by a central angle that measures θ radians in a circle with radius r.

MATH REFERENCE FACT

The number of radians of arc in a semicircle is π and the number of radians of arc in a circle is 2π.

Converting Between Radians and Degrees

Since the degree measure of arc of a circle is 360°,

$$2\pi \text{ radians} = 360° \text{ so } \pi \text{ radians} = \frac{360°}{2} = 180°$$

TIP

Memorize the radian equivalents of 30°, 45°, and 90°:

- **$30° = \frac{\pi}{6}$ radians**
- **$45° = \frac{\pi}{4}$ radians**
- **$90° = \frac{\pi}{2}$ radians**

This relationship provides a way of converting between radian and degree measures:

- To change from degrees to radians, multiply the number of degrees by $\frac{\pi}{180°}$. For example,

$$60° = \overset{3}{\cancel{60°}} \times \frac{\pi}{\cancel{180°}} = \frac{\pi}{3} \text{radians}$$

- To change from radians to degrees, multiply the number of radians by $\frac{180°}{\pi}$. For example,

$$\frac{7}{12}\pi \text{ radians} = \frac{7\cancel{\pi}}{\cancel{12}} \times \frac{\overset{15°}{\cancel{180}}°}{\cancel{\pi}} = 105°$$

If you memorize the radian equivalents of 30°, 45°, and 90°, then you can use these values to quickly figure out the radian equivalents of their multiples. For example,

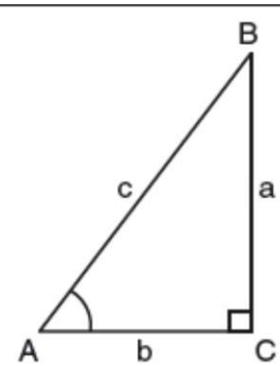
- $60° = 2 \times 30° = 2 \times \frac{\pi}{6} = \frac{\pi}{3}$ radians
- $150° = 5 \times 30° = 5 \times \frac{\pi}{6} = \frac{5\pi}{6}$ radians
- $270° = 3 \times 90° = 3 \times \frac{\pi}{2} = \frac{3\pi}{2}$ radians
- $315° = 7 \times 45° = 7 \times \frac{\pi}{4} = \frac{7\pi}{4}$ radians

> **TIP**
>
> **Angles measured in radians are not always expressed in terms of π. An angle of 2 radians measures $2 \times \frac{180°}{\pi} \approx 114.6°$.**

RIGHT TRIANGLE TRIGONOMETRY

The three basic trigonometric functions of **sin**e, **cos**ine, and **tan**gent are defined in Table 6.3.

Table 6.3 The Three Basic Trigonometric Functions

Basic Three Trigonometric Functions	Quotient Relationships	
$\sin A = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}} = \frac{a}{c}$		
$\cos A = \frac{\text{leg adjacent } \angle A}{\text{hypotenuse}} = \frac{b}{c}$	$\frac{\sin A}{\cos A} = \tan A$	
$\tan A = \frac{\text{leg opposite } \angle A}{\text{leg adjacent } \angle A} = \frac{a}{b}$		

Cofunction Relationships

Two angles are complementary if their measures add up to $90° \left(= \frac{\pi}{2} \text{radians}\right)$. The sine and cosine functions have equal values when their angles are complementary. For example, sin 50° = cos 40° and $\sin 50° = \cos 40°$ and $\cos\frac{\pi}{3} = \sin\frac{\pi}{6}$. In general,

- If x is an acute angle measured in degrees, then

$$\sin x = \cos(90 - x) \text{ or, equivalently, } \cos x = \sin(90 - x)$$

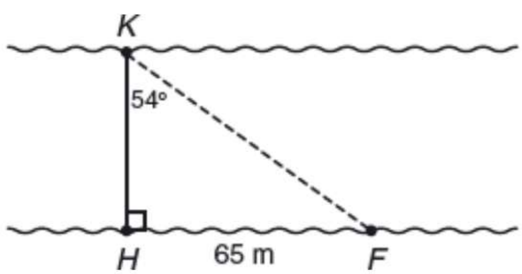
- If x is an acute angle measured in radians, then

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \text{ or, equivalently, } \cos\frac{\pi}{2} = \sin\left(\frac{\pi}{2} - x\right)$$

INDIRECT MEASUREMENT

Trigonometric functions are particularly useful when it is necessary to calculate the measure of a side or an angle of a right triangle that may be difficult, if not impossible, to measure directly.

➥ Example



To determine the distance across a river, as shown in the figure above, a surveyor marked two points on one riverbank, H and F, 65 meters apart. She also marked one point, K, on the opposite bank such that $\overline{KH} \perp \overline{HF}$. If $\angle K = 54°$ and x is the width of the river, which of the following equations could be used to find x?

(A) $\tan 54° = \frac{x}{65}$

(B) $\sin 36° = \frac{x}{65}$

(C) $\tan 36° = \frac{x}{65}$

(D) $\sin 54° = \frac{x}{65}$

Solution

Represent the width of the river, *KH*, by *x*.

Because the problem involves the sides opposite and adjacent to the given angle, use the tangent ratio:

$$\tan\angle K = \frac{\text{opposite side }(HF)}{\text{adjacent side }(KH)}$$

$$\tan 54° = \frac{65}{x}$$

Since this is not one of the answer choices, consider the other acute angle of the right triangle at *F*, which measures 90° − 54° = 36°:

$$\tan\angle F = \frac{KH}{HF}$$

$$\tan 36° = \frac{x}{65}$$

The correct choice is **(C)**.

ANGLES IN STANDARD POSITION

Trigonometric functions of angles greater than $\frac{\pi}{2}$ radians (= 90°) or less than 0 radians (= 0°) can be given meaning by placing angles in *standard position* in the *xy*-plane. An angle is in **standard position** when its vertex is at the origin in the *xy*-plane and one of its sides, called the **initial side**, remains fixed on the *x*-axis. The side of the angle that rotates about the origin is called the **terminal side** of the angle.

- If the terminal side of an angle rotates in a counterclockwise direction, as shown in Figure 6.24a, the angle is *positive*.
- If the terminal side of an angle rotates in a clockwise direction, as shown in Figure 6.24b, the angle is *negative*.

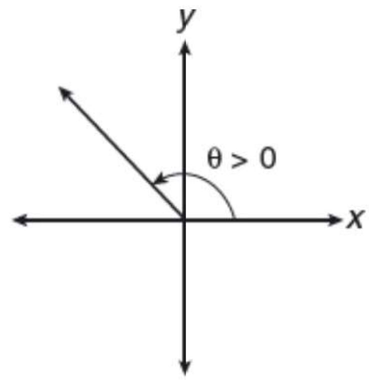


Figure 6.24a Positive angle

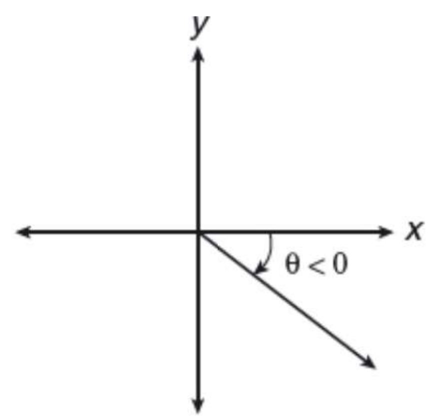


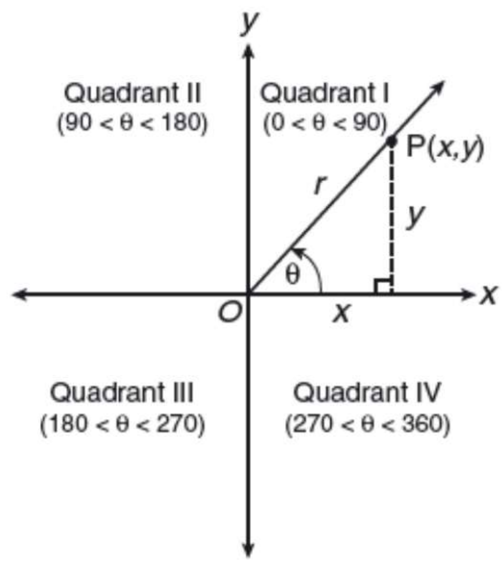
Figure 6.24b Negative angle

DEFINING TRIGONOMETRIC FUNCTIONS USING COORDINATES

If $P(x, y)$ is any point on the terminal side of an angle in standard position and r is the distance of point P from the origin, then trigonometric functions can be defined in terms of x, y, and r as shown in Table 6.5. If a perpendicular is drawn from $P(x, y)$ to the x-axis, a right triangle is formed in which x, y, and r are related by the Pythagorean equation $x^2 + y^2 = r^2$.

> **TIP**
>
> **If $P(x, y)$ is a point on the terminal side of an angle θ in standard position, then the x-coordinate of point P is $r\cos\,\theta$ and the y-coordinate is $r\sin\,\theta$ where r, the distance from the origin to point P, is $\sqrt{x^2 + y^2}$.**

Table 6.5 Coordinate Definitions of the Basic Trigonometric Functions

Coordinate Definitions	Standard Position of Angle θ In the xy-plane
If $P(x, y)$ is on the terminal side of an angle, θ in standard position then: ■ $\sin\theta = \frac{y}{r}$ ■ $\cos\theta = \frac{x}{r}$ ■ $\tan\theta = \frac{y}{x}$ where $x^2 + y^2 = r^2$	

SIGNS OF TRIGONOMETRIC FUNCTIONS IN THE FOUR QUADRANTS

The signs of trigonometric functions of angle θ depend on the signs of x and y in the particular quadrant in which the terminal side of θ lies, as shown in Table 6.6. For example, in Quadrant II, $x < 0$ and $y > 0$ so $\tan\theta = \frac{y}{x} = \frac{+}{-} = -$. Quadrant I is the only quadrant in which all of the trigonometric functions are positive at the same time.

> **TIP**
>
> **The first letter of each word of the phrase "*A*ll *S*tudents *T*ake *C*alculus" can help you remember the quadrants in which a trigonometric function is positive.**

Table 6.6 Quadrants in Which Trigonometric Functions Are Positive

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
All are +	Sine is +	Tangent is +	Cosine is +

➡ Example

If $P(-3, 4)$ is a point on the terminal side of angle θ, what is the value of cos θ?

(A) $-\frac{3}{4}$

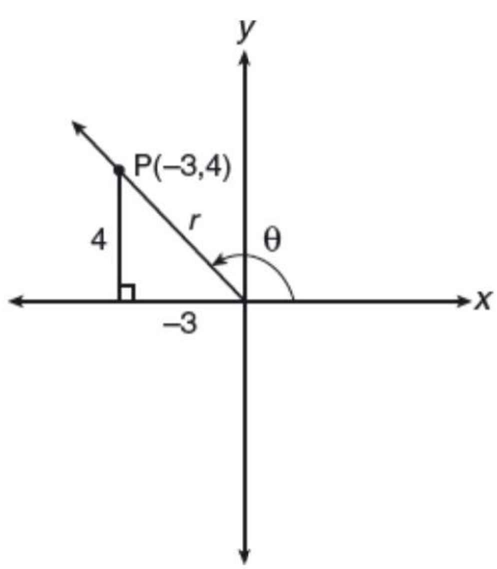
(B) $-\frac{3}{5}$

(C) $\frac{3}{4}$

(D) $\frac{4}{5}$

Solution

Since $x = -3$ and $y = +4$, the terminal side of the angle θ lies in Quadrant II:

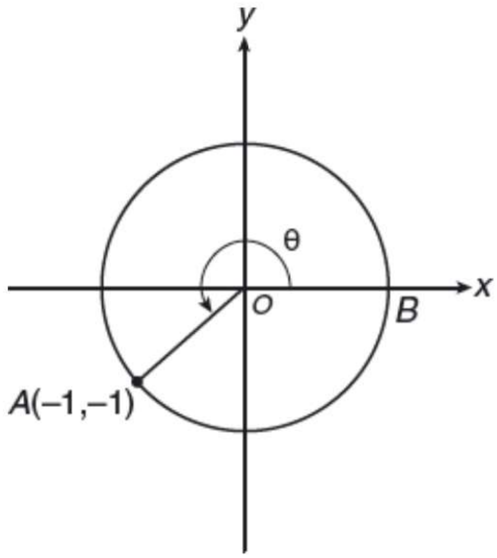


- The lengths of the sides of the right triangle form a 3-4-5 Pythagorean triple so $r = 5$.
- Use the coordinate definition of cosine:

$$\cos\theta = \frac{x}{r} = -\frac{3}{5}$$

The correct choice is **(B)**.

➥ Example

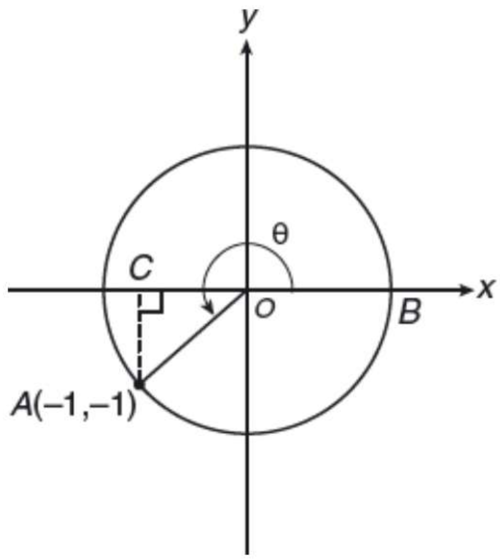


In the *xy*-plane above, *O* is the center of the circle, and the measure of angle θ is $k\pi$ radians. If $0 \le \theta \le 2\pi$, what is the value of *k* ?

Solution

Since $(-1, -1) = (x, y)$, $\frac{y}{x}$ so $\triangle AOC$ is a 45°-45° right triangle with $m\angle AOC =$ 45. Hence, θ measures 225° or, equivalently, $\frac{5}{4}\pi$ radians.

Grid-in **5/4**



Angles Greater Than 2π Radians

An angle of 410° exceeds one complete rotation by 410° − 360° = 50° so its terminal side will lie in Quadrant I and form an angle of 50° with the positive x-axis. A trigonometric function of an angle greater than 2π radians or less than 0 radians can be written as the same function of an angle between 0 and 2π radians by subtracting or adding a multiple of 2π:

- $\sin 410° = \sin(410° - 360°) = \sin 50°$
- $\cos 870° = \cos(870° - 720°) = \cos 150°$
- $\tan\frac{9\pi}{4} = \tan\left(\frac{9\pi}{4} - 2\pi\right) = \tan\frac{\pi}{4}$
- $\cos\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3} + 2\pi\right) = \cos\frac{5\pi}{3}$

"Reducing" Angles of Trigonometric Functions

The **reference angle**, denoted by θ', is the acute angle formed by the terminal side of the angle in standard position and the x-axis as illustrated in Figure 6.25.

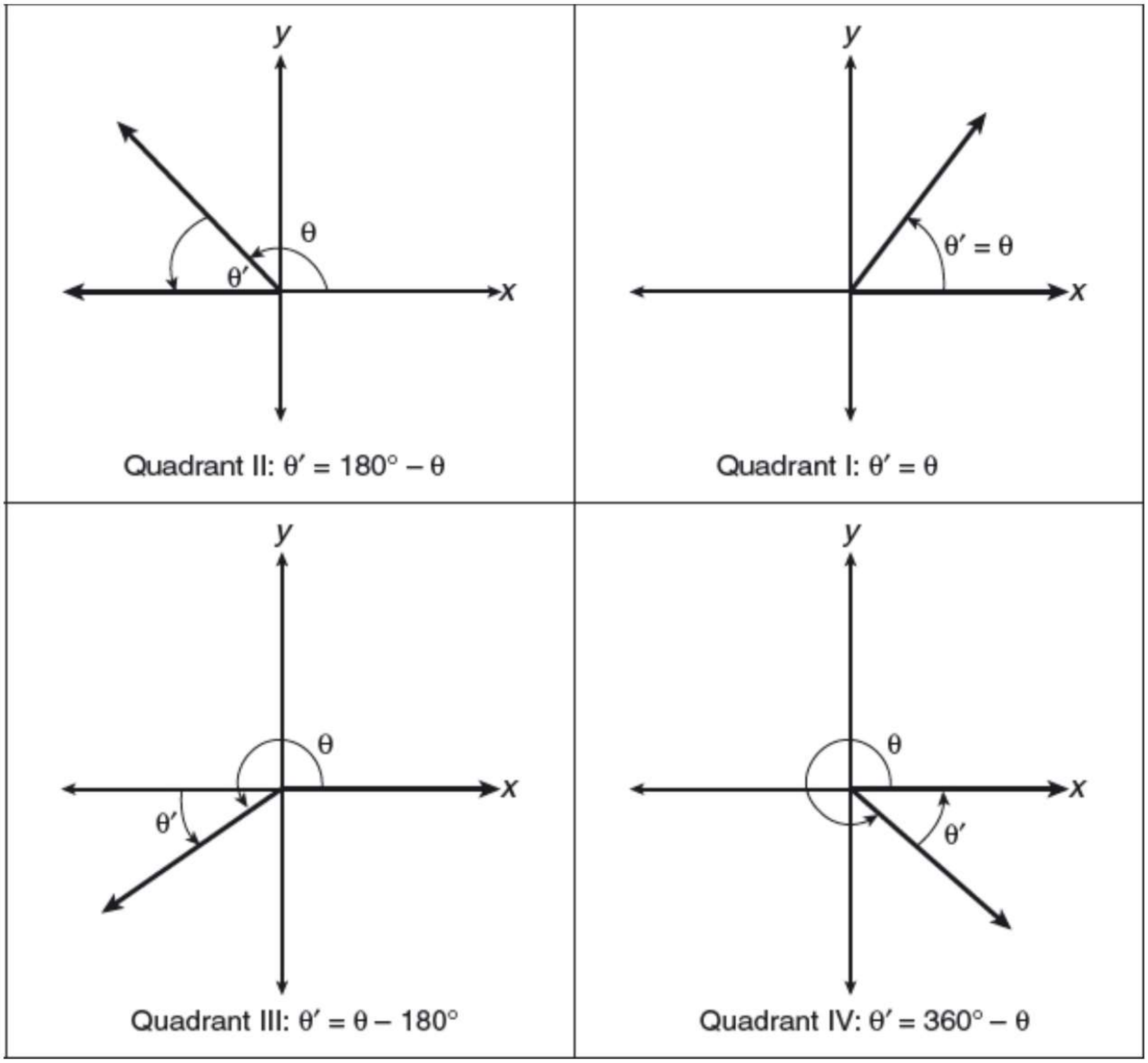
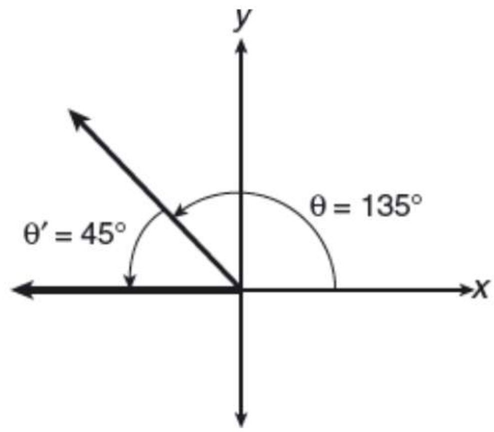


Figure 6.25 Reference angle θ' in the four quadrants

The trigonometric function of any angle θ can be expressed as either plus or minus the same trigonometric function of its *reference angle*. To express cos 135° as a function of its reference angle,



- Locate the reference angle, θ', as shown in the accompanying figure, and find its measure:

$$\theta' = 180 - 135 = 45$$

- Determine the sign of the cosine function in the quadrant in which θ' is located. Since cosine is negative in Quadrant II,

$$\cos 135° = -\cos 45°$$

Similarly, because sine is positive in Quadrant II while tangent is negative,

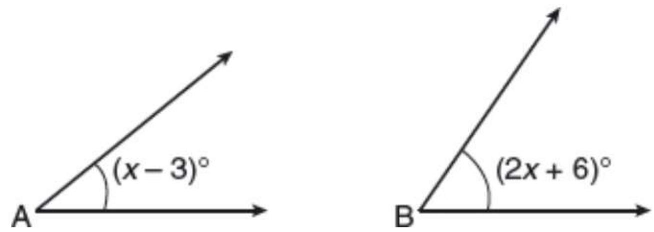
$$\sin 135° = \sin 45° \text{ and } \tan 135° = -\tan 45°$$

You should draw your own diagram and confirm that

- $\sin 310° = -\sin 50°$
- $\cos 670° = \cos (670 - 360)° = \cos 50°$
- $\tan 880° = \tan (880 - 720)° = -\tan 20°$
- $\tan\frac{5\pi}{4} = \tan\frac{\pi}{4}$.
- $\sin (-140°) = \sin 220° = -\sin 40°$
- $\cos\frac{10\pi}{3} = \cos\left(\frac{10\pi}{3} - 2\pi\right) = -\cos\frac{\pi}{3}$

LESSON 6-5 TUNE-UP EXERCISES

Multiple-Choice



1. If in the figure above $\frac{\sin A}{\cos B} = 1$, then $x =$

(A) 6
(B) 26
(C) 29
(D) 59

2. By law, a wheelchair service ramp may be inclined no more than 4.76°. If the base of a ramp begins 15 feet from the base of a public building, which equation could be used to determine the maximum height, h, of the ramp where it reaches the building's entrance?

(A) $h = 15 \sin 4.76°$
(B) $h = \frac{15}{\sin 4.76°}$
(C) $h = \frac{\tan 4.76°}{15}$
(D) $h = 15 \tan 4.76°$

3. What is the number of radians through which the minute hand of a clock turns in 24 minutes?

(A) 0.2π
(B) 0.4π
(C) 0.6π
(D) 0.8π

4. If $x = 1.75$ radians, then the value of $\cos x$ is closest in value to which of the following?

(A) $-\cos 1.39$

(B) cos 4.89
(C) cos 4.53
(D) −cos 0.18

5. If $\sin\frac{2}{9}\pi = \cos x$, then $x =$

(A) $\frac{7}{9}\pi$

(B) $\frac{5}{18}\pi$

(C) $\frac{\pi}{3}$

(D) $\frac{13}{18}\pi$

6. The bottom of a pendulum traces an arc 3 feet in length when the pendulum swings through an angle of $\frac{1}{2}$ radian. What is the number of feet in the length of the pendulum?

(A) 1.5
(B) 6
(C) $\frac{1.5}{\pi}$
(D) 6π

7. What is the radian measure of the smaller angle formed by the hands of a clock at 7 o'clock?

(A) $\frac{\pi}{2}$

(B) $\frac{2\pi}{3}$

(C) $\frac{5\pi}{6}$

(D) $\frac{7\pi}{6}$

8. A wheel has a radius of 18 inches. The distance, in inches, the wheel travels when it rotates through an angle of $\frac{2}{5}\pi$ radians is closest to which value?

(A) 45

(B) 23
(C) 13
(D) 11

9. A wedge-shaped piece is cut from a circular pizza. The radius of the pizza is 14 inches and the angle of the pointed end of the pizza measures 0.35 radians. The number of inches in the length of the rounded edge of the crust is closest to which value?

(A) 4.0
(B) 4.9
(C) 5.7
(D) 7.5

I. $x = y$
II. $2(x + y) = \pi$
III. $\cos x = \sin y$

10. If $0 < x, y < \frac{\pi}{2}$, and $\sin x = \cos y$, then which of the statements above must be true?

(A) I and II only
(B) II and III only
(C) II only
(D) III only

11. If $\cos \theta = -\frac{3}{4}$ and $\tan \theta$ is negative, the value of $\sin \theta$ is

(A) $-\frac{4}{5}$

(B) $-\frac{\sqrt{7}}{4}$

(C) $\frac{4\sqrt{7}}{7}$

(D) $\frac{\sqrt{7}}{4}$

12. If $\cos A = \frac{4}{5}$ and $\angle A$ is *not* in Quadrant I, what is the value of $\sin A$?

(A) –0.6
(B) –0.2

(C) 0.6
(D) 0.75

13. If $\sin A = b$, what is the value of the product $\sin A \cdot \cos A \cdot \tan A$ in terms of b?

(A) 1
(B) $\frac{1}{b}$
(C) b
(D) b^2

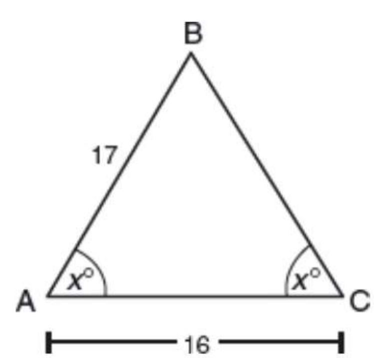
14. The equatorial diameter of the earth is approximately 8,000 miles. A communications satellite makes a circular orbit around the earth at a distance of 1,600 miles from the earth. If the satellite completes one orbit every 5 hours, how many miles does the satellite travel in 1 hour?

(A) $1{,}120\pi$
(B) $1{,}940\pi$
(C) $2{,}240\pi$
(D) $2{,}560\pi$

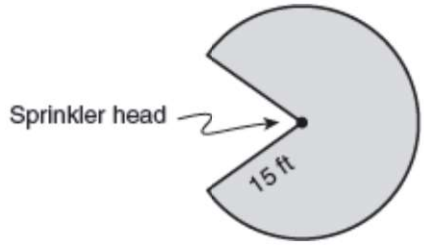
15. A rod 6 inches long is pivoted at one end. If the free end of the rod rotates in a machine at the rate of 165 revolutions per minute, what is the total distance, in inches, traveled by the end of the rod in one *second*?

(A) 14.5π
(B) 16.5π
(C) 29π
(D) 33π

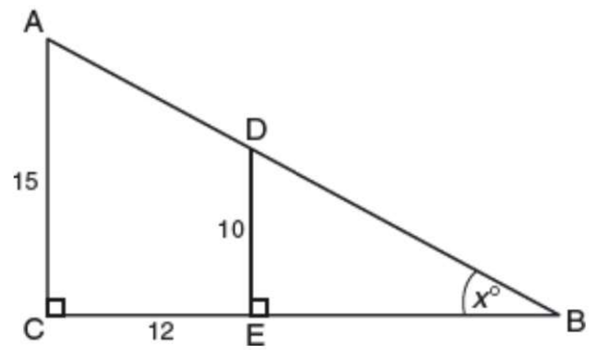
Grid-In



1. In the figure above, what is the value of $\sin A - \cos A$?



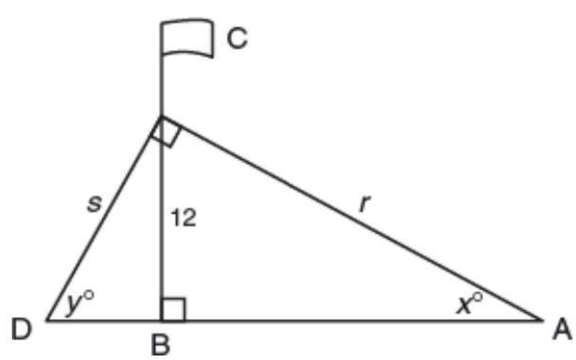
2. A lawn sprinkler sprays water in a circular pattern at a distance of 15 feet from the sprinkler head which rotates through an angle of $\frac{5\pi}{3}$ radians, as shown by the shaded area in the diagram above. What is the area of the lawn, to the *nearest square foot*, that receives water from this sprinkler?



Note: Figure not drawn to scale.

3. In the figure above, angles *ACB* and *DEB* are right angles, $AC = 15$, $CE = 12$, and $DE = 10$. What is the value of $\cos x$?

For **Questions 4 and 5** refer to the diagram.



A flagpole that stands on level ground. Two cables, r and s, are attached to the pole at a point 12 feet above the ground and form a right angle with each other. Cable r is attached to the ground at a point that makes $\tan x = 0.75$.

4. What is the value of $\cos x$?

5. What is the sum of the lengths of cables r and s?

LESSON 6-6 THE UNIT CIRCLE

OVERVIEW

A **unit circle** is a circle with a radius of 1. The coordinates of a point on the unit circle centered at the origin in the *xy*-plane can be expressed in terms of cosine and sine.

THE UNIT CIRCLE

If the terminal side of angle θ intersects a unit circle in the *xy*-plane at $P(x, y)$, as shown in Figure 6.26, then

- $P(x, y) = P(\cos\theta, \sin\theta)$ since $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{1} = y$.
- $\sin^2\theta + \cos^2\theta = 1$ because $x^2 + y^2 = 1$.

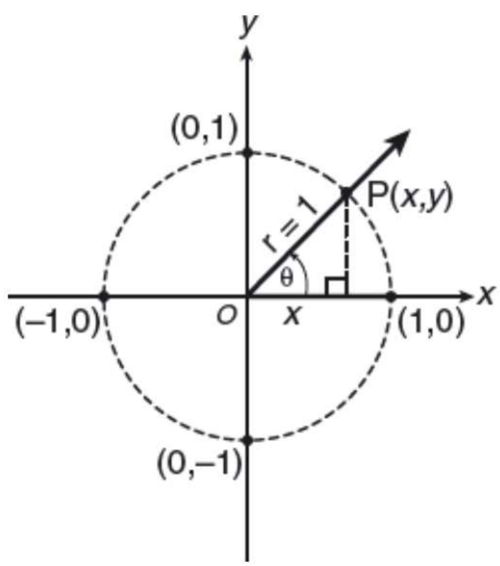


Figure 6.26 The unit circle

MATH REFERENCE FACT

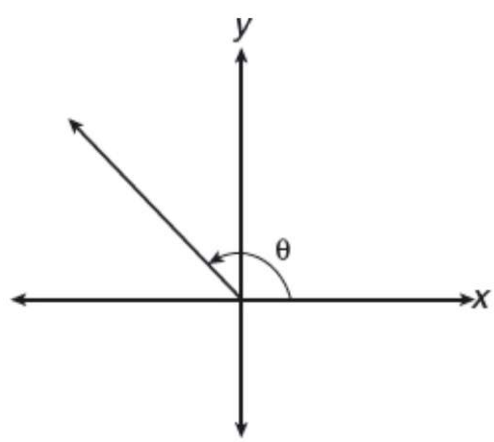
The maximum and minimum values of $\sin\theta$ and $\cos\theta$ occur when the terminal side of angle θ coincides with a coordinate axis.

RANGE OF VALUES OF SINE AND COSINE

Since the y-values of the sine and cosine functions are coordinates of points on the unit circle, $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$ where

- $\sin\frac{\pi}{2} = 1$ and $\sin\frac{3\pi}{2} = -1$.
- $\cos 0 = \cos 2\pi = 1$ and $\cos\pi = -1$.

➡ Example



In the figure above, if $\cos\theta = -0.36$, what is the value of $\sin\theta$ to the *nearest hundredth*?

(A) 0.64
(B) 0.80
(C) 0.93
(D) −0.93

Solution

Use the relationship $\sin^2\theta + \cos^2\theta = 1$ to solve for $\sin\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + (0.36)^2 = 1$$
$$\sin^2\theta = 1 - 0.1296$$
$$\sin\theta = \pm\sqrt{0.8704} \approx \pm 0.93$$

Since sine is positive in Quadrant II, the correct choice is **(C)**.

➡ Example

If sin $w = a$ and $\frac{\pi}{2} < w < \pi$, what is tan w in terms of a?

(A) $\dfrac{a}{\sqrt{1-a^2}}$

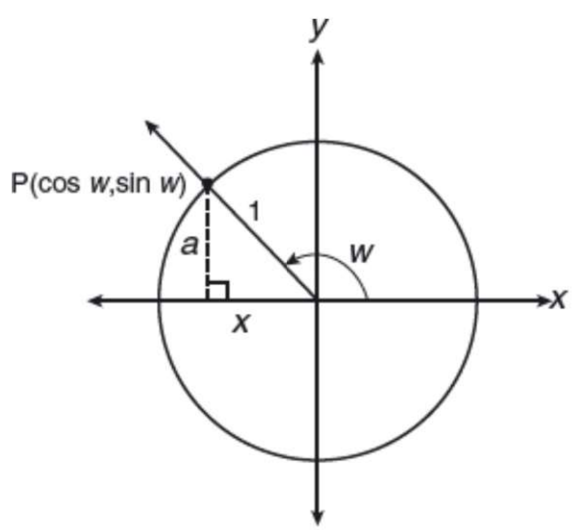
(B) $\dfrac{-a}{\sqrt{1-a^2}}$

(C) $\dfrac{1}{1-a}$

(D) $\dfrac{-a}{1-a}$

Solution

The terminal side of angle w in standard position intersects the unit circle in Quadrant II at a point P whose x-coordinate is cos w and y-coordinate is sin w:



- Form a right triangle by dropping a perpendicular to the *x*-axis as shown in the accompanying figure. The lengths of the legs of the right triangle correspond to the coordinates of point *P*.
- Use the Pythagorean theorem to express *x* (= cos *w*) in terms of *a*:

$$x^2 + a^2 = 1$$
$$x^2 = 1 - a^2$$
$$x = -\sqrt{1-a^2} = \cos w \leftarrow x \text{ is negative in Quadrant II}$$

- Find tan *w*:

$$\tan w = \frac{\sin w}{\cos w} = \frac{a}{-\sqrt{1-a^2}} = \frac{-a}{\sqrt{1-a^2}}$$

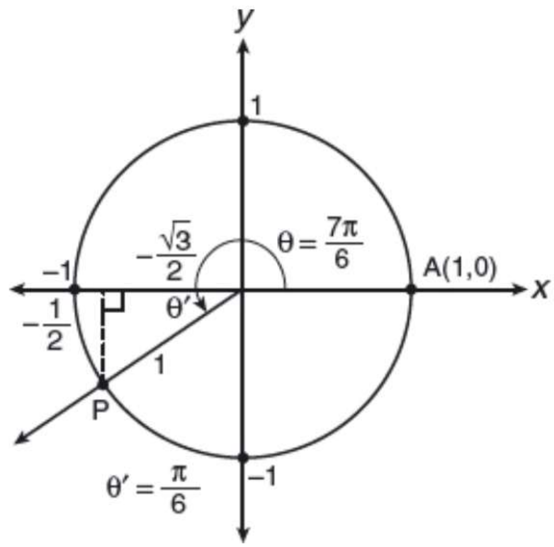
The correct choice is **(B)**.

➡ Example

Point *P* is on the unit circle with center at *O* and point *A* is the point at which the unit circle intersects the positive *x*-axis. If angle *AOP* measures $\frac{7\pi}{6}$ radians, what are the coordinates of point *P*?

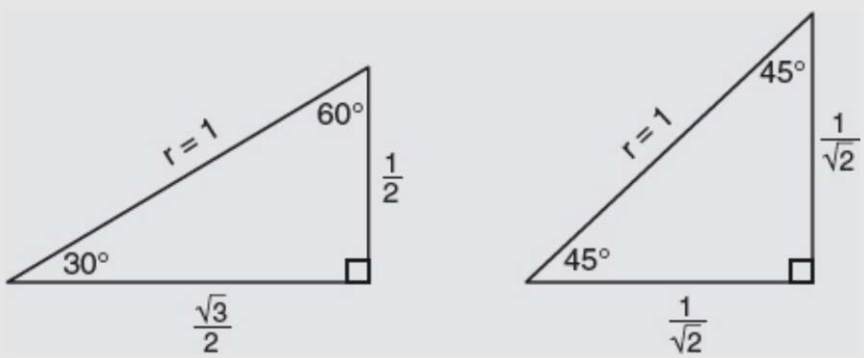
Solution

If $\theta = \frac{7\pi}{6}$, then θ is in Quadrant III, and the reference angle is $\frac{\pi}{6}$(=30°). The right triangle that contains the reference angle is a 30-60 right triangle with a hypotenuse of 1. Since *x* and *y* are both negative in Quadrant III, $x = -\frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$ so the coordinates of point *P* are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



MATH REFERENCE FACT

When finding the coordinates of points on rays whose angles of rotation within the unit circle are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, use these special right triangle relationships:



➡ Example

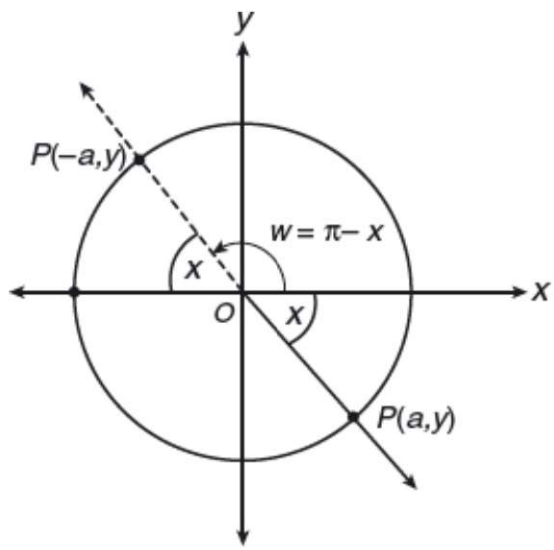
If $\cos x = a$, $\cos w = -a$, and $-\frac{\pi}{2} < x < 0$, which of the following is a possible value of w?

(A) $\pi - x$
(B) $x - \pi$
(C) $2\pi - x$

(D) $x + 2\pi$

Solution

Since $-\frac{\pi}{2} < x < 0$ and $\cos x = a$, locate point $P(a, y)$ in Quadrant IV with reference angle x. We need to find the angle, w, that the ray through $P(-a, y)$ makes with the positive x-axis. Locate $P(-a, y)$ in Quadrant II on the ray opposite OP. Since the reference angle is also x, the angle of rotation, w, is $\pi - x$ as shown in the accompanying figure.



The correct choice is **(A)**.

PERIODIC FUNCTIONS

The sine and cosine functions are *periodic* functions. A function is **periodic** if its values repeat at regular intervals. The sine and cosine functions each have a period of 2π radians since each time 2π is added to an angle, or subtracted from an angle, we go around the unit circle and return to the same point. For example, $\sin(2\pi + x) = \sin x$ and $\cos(x - 2\pi) = \cos x$.

SYMMETRY IN THE UNIT CIRCLE

In Figure 6.27, P' is the reflection of P in the x-axis.

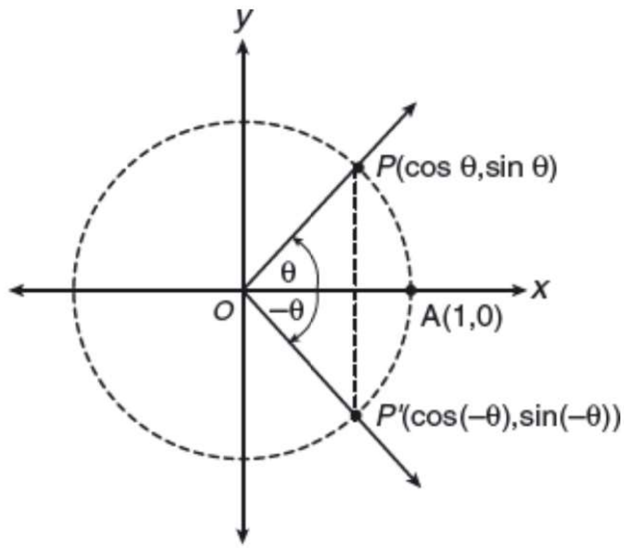


Figure 6.27 Symmetry in the unit circle

- The angle ray OP' makes with the terminal side is equal in measure but clockwise in rotation compared to the angle ray OP makes with the terminal side. Thus, the coordinates of point P' are (cos $(-\theta)$, sin $(-\theta)$).
- Because P' is the reflection of P in the x-axis, it has the same x-coordinate as point P but the opposite y-coordinate:

$$\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta$$

- Since $\tan\theta$ is the quotient of $\sin\theta$ divided by $\cos\theta$,

$$\tan(-\theta) = -\tan\theta$$

TIP

You should memorize:

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$

GENERAL REDUCTION RELATIONSHIPS

For SAT test purposes, to determine whether a general relationship such as $\sin(x + \pi) = -\sin x$ is true or false, use a convenient test value for x. If

$x = 30^\circ \left(= \frac{\pi}{6}\right)$, then $x + \pi$ is a Quadrant III

angle and the reference angle is 30° as illustrated in Figure 6.28.

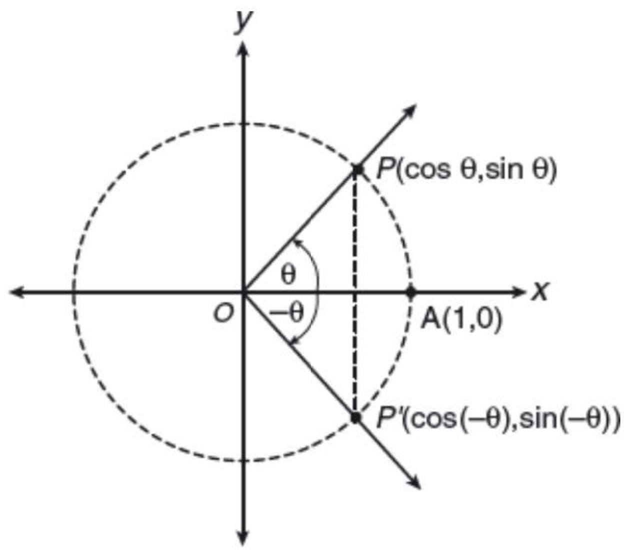


Figure 6.28 Illustrating sin 210° = −sin 30°

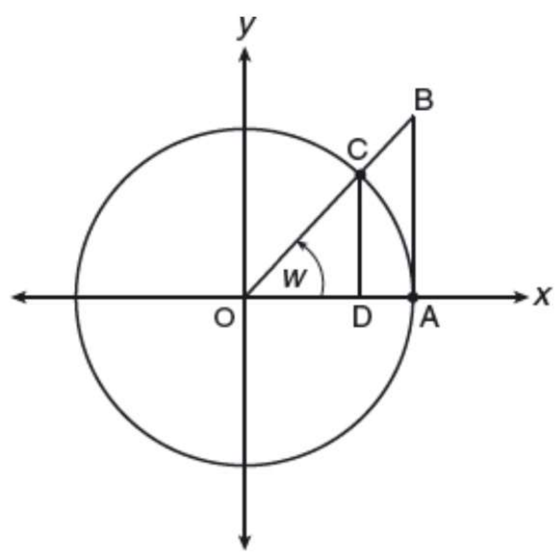
Sine is negative in Quadrant III so sin (30° + 180°) = sin 210°= −sin 30°. You can then make the generalization that $\sin(x + \pi) = -\sin x$. Similarly, $\sin(x - \pi) = -\sin x$.

LESSON 6-6 TUNE-UP EXERCISES

Multiple-Choice

1. The path traveled by a roller coaster is modeled by the equation $y = 27 \sin 13x + 30$ where y is measured in meters. What is the number of meters in the maximum altitude of the roller coaster?

   (A) 13
   (B) 27
   (C) 30
   (D) 57



2. The unit circle above has radius $\overline{OC}$, angle AOB measures w radians, $\overline{BA}$ is tangent to circle O at A, and $\overline{CD}$ is perpendicular to the x-axis. The length of which line segment represents $\sin w$?

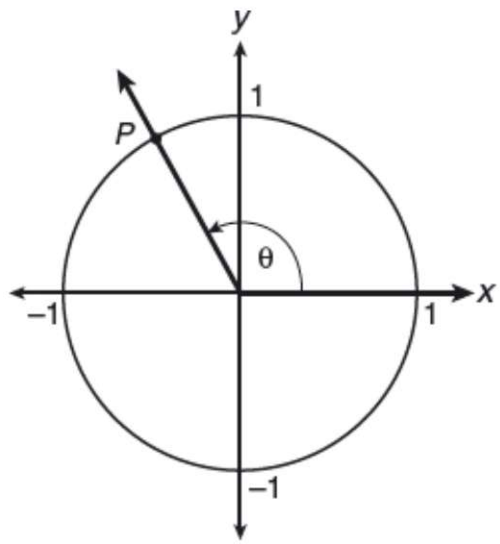
   (A) $\overline{OD}$
   (B) $\overline{CD}$
   (C) $\overline{AB}$
   (D) $\overline{OB}$

3. If x is an acute angle, which expression is *not* equivalent to $\cos x$?

   (A) $-\cos(-x)$

   (B) $\left(\frac{\pi}{2} - x\right)$

(C) $-\cos(x+\pi)$
(D) $\cos(x-2\pi)$



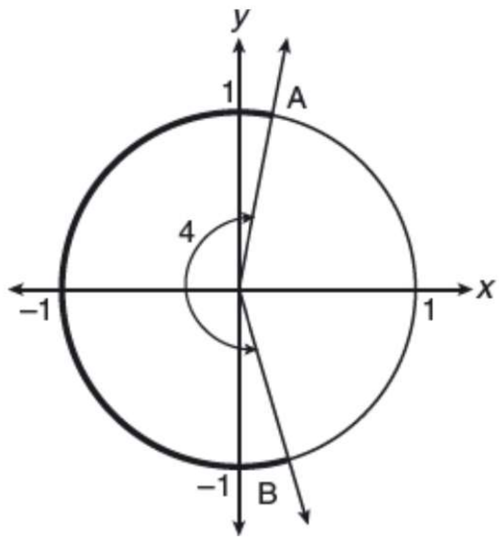
4. In the figure above, θ is an angle in standard position and its terminal side passes through the point $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. What is a possible value of θ?

(A) $\frac{2}{3}\pi$

(B) $\frac{5}{6}\pi$

(C) $\frac{7}{6}\pi$

(D) $\frac{4}{3}\pi$



5. In the unit circle above, an angle that measures 4 radians intercepts arc *AB*. What is the length of major arc *AB*?

(A) $\frac{\pi}{2}$

(B) 4

(C) $\frac{\pi + 2}{4}$

(D) $\frac{4}{\pi}$

6. If θ is an angle in standard position and its terminal side passes through the point $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle, then a possible value of θ is

(A) $\frac{7\pi}{6}$

(B) $\frac{4\pi}{3}$

(C) $\frac{5\pi}{3}$

(D) $\frac{11\pi}{6}$

7. What are the coordinates of the image of the point (1, 0) on the terminal side of an angle after a clockwise rotation of $\frac{\pi}{6}$ radians?

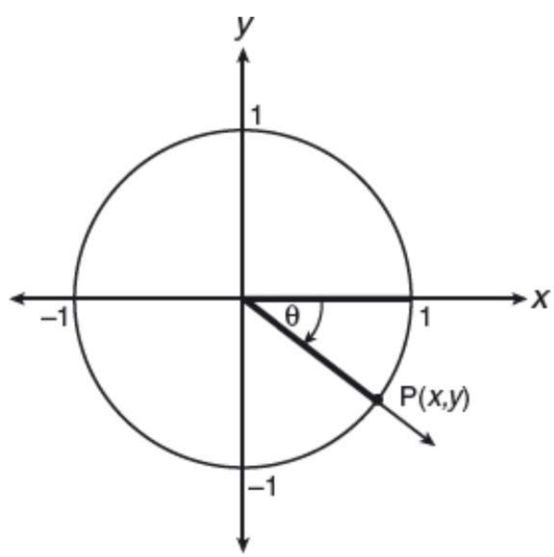
(A) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
(B) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
(C) $\left(-\frac{\sqrt{3}}{2}, 1\right)$
(D) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

8. What are the coordinates of the image of the point (1, 0) on the terminal side of an angle after a counterclockwise rotation of $\frac{3}{4}\pi$ radians?

(A) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
(B) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
(C) $\left(-\sqrt{2}, 1\right)$
(D) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

9. Which of the following expressions is equivalent to $\frac{\sin^2 x}{1+\cos x}$?

(A) $1 - \sin x$
(B) $1 - \cos x$
(C) $\sin x + \cos x$
(D) $\sin x - \cos x$



10. In the unit circle above, the ordered pair (x, y) represents a point P where the terminal side intersects the unit circle, as shown in the accompanying figure. If $\theta = -\frac{\pi}{3}$ radians, what is the value of y?

(A) $-\frac{\sqrt{3}}{2}$

(B) $-\frac{\sqrt{2}}{2}$

(C) $-\sqrt{3}$

(D) $-\frac{1}{2}$

11. If x is a positive acute angle and $\cos x = a$, an expression for $\tan x$ in terms of a is

(A) $\frac{1-a}{a}$

(B) $\sqrt{1-a^2}$

(C) $\frac{\sqrt{1-a^2}}{a}$

(D) $\frac{1}{1-a}$