Pre-Calculus: Course Study Guide

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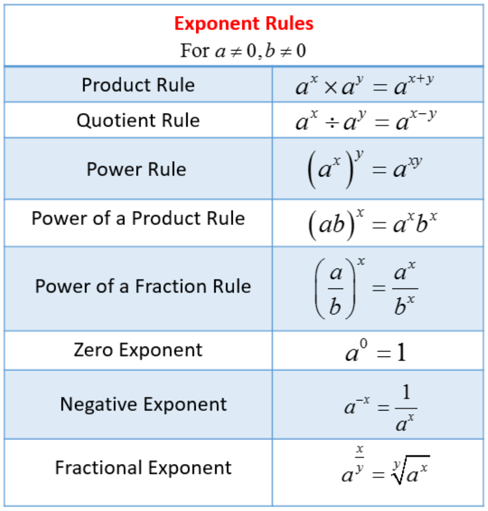
❖ Prerequisites: Fundamental Concepts of Algebra

Algebraic Expressions and Real Numbers

- **Natural Numbers**: positive numbers
- **Whole Numbers**: number without fractions
- **Integers**: whole numbers and their opposite
- **Simple Fraction**: integer/natural
- **Absolute Value**: distance from 0
- **Evaluate** -> Expressions
  - 2x+3; when x=3
- **Solve** -> Equations
  - 11= 2x+3
- **Intersection (∩):** in both brackets at the same time
  - Ex: $A = \{1,3,4\} B = \{3,4,5,6\}$= A∩B: $\{3,4\}$
- **Union (∪):** joining of things, join what's inside
- { }: empty set; ∅: "null set" = no values



Exponent Rules For $a \neq 0, b \neq 0$	
Product Rule	$a^x \times a^y = a^{x+y}$
Quotient Rule	$a^x \div a^y = a^{x-y}$
Power Rule	$(a^x)^y = a^{xy}$
Power of a Product Rule	$(ab)^x = a^x b^x$
Power of a Fraction Rule	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
Zero Exponent	$a^0 = 1$
Negative Exponent	$a^{-x} = \frac{1}{a^x}$
Fractional Exponent	$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

Radicals

- **Radical**: the main/primary root of what's underneath
  - $\sqrt{36}$ = square root
  - $\sqrt[3]{8}$ = cube root
    - $\sqrt{4x^2} = 2|x|$
      - the absolute value keeps the x from becoming negative
      - **IMPORTANT**: The only time you need the absolute value is when you take an **even root** of an **even power** and get an **odd result**. (EEO)
  - Examples

    - $\sqrt{49x^2y^4}=7|x^3|y^2$
    - $2\sqrt[3]{5}+\sqrt[3]{5}=3\sqrt[3]{5}$

Factoring Formulas

- **$(a+b)^2$ =** $a^2+2ab+b^2$
    - $49x^2+126x+81=(7x+9)^2$
- **$(a+b)(a-b)$ =** a^2-b^2**(difference of 2 squares)**
- a^2+b^2**= sum of 2 squares (can't factor) = prime**
- $a^3-b^3=(a-b)\,(a^2+ab+b^2)$**= can never factor further than this!**
- a^3+b^3 **=sum of 2 cubes**
    - $(a+b)\,(a^2-ab+b^2)$

Rational Expressions

- **Simplifying Rational Expressions**
    - 1) Factor the numerator and denominator completely
    - 2) Divide both the numerator and denominator by any common factors
        - Ex. $\frac{x^3+x^2}{x+1}=\frac{x^2(x+1)}{x+1}=\frac{x^2(x+1)}{(x+1)}$- *cancel (x+1)*
            - $=x^2, x\neq -1$
- **Multiplying Rational Expressions**
    - 1) **Factor** all numerators and denominators completely
    - 2) **Divide** numerators and denominators by common factors
    - 3) **Multiply** the remaining factors in the numerators and multiply the remaining factors in the denominators

$$\frac{3x}{8y^2}\times\frac{y}{12}=\frac{{}^{1}\cancel{3}x}{8y^{\cancel{2}}{}_{1}}\times\frac{\cancel{y}}{\cancel{12}{}_{4}}$$

$$=\frac{x}{32y}$$

- **Dividing Rational Expressions**
    - 1) Find the answer by **inverting** the second divisor and multiplying using the steps above

$$\frac{7p}{p+2}\div\frac{p+2}{p}=\frac{7p}{p+2}\times\frac{p}{p+2}$$

$$=\frac{7p^2}{(p+2)^2}$$

- **Adding/Subtracting Rational Expressions**

That Have Different Denominators

- 1) Find the **LCD** of the rational expression
- 2) Rewrite each rational expression as an equivalent expression whose denominator is the LCD.
- 3) Add/Subtract numerators by placing the resulting expression over the LCD.
- 4) If possible, simplify.

Finding Domain

- Ex. $\frac{(x-3)(2x+1)}{(x+1)(x-3)} = \frac{2x+1}{x+1}$ =**Domain** = $(-\infty, -1) \cup (-1, \infty)$

Equations

- **Linear Equation**: an equation that is written in the form **ax+b=0** where a and b are real numbers and a$\neq 0$
- **Solving Linear Equation**
  - 1) Simplify both expressions
  - 2) Group all variable terms on one side and constant terms on the other
  - 3) Isolate the variable and solve

$$2(x+2)-5=3(x+1)$$
$$2x-1=3x+3$$
$$-x-1=3$$
$$-x=4$$
$$x=-4$$

- **Solving Rational Equations**
  - 1) Find the least common denominator
  - 2) Use the distributive property and divide out common factors
  - 3) Simplify

$$3x\cdot\left(\frac{5}{x}-\frac{1}{3}\right)=3x\cdot\left(\frac{1}{x}\right)$$
$$3x\cdot\frac{5}{x}-3x\cdot\frac{1}{3}=3x\cdot\frac{1}{x}$$
$$15-x=3$$
$$-x=-12$$
$$x=12$$

- **Solving Equation Involving Absolute Value**
  - Isolate absolute value
  - Split into 2 different equations
  - Solve for both

$$|5-2x|-11=0$$
$$|5-2x|=11$$

$$5-2x=11 \qquad 5-2x=-11$$
$$-2x=6 \qquad -2x=-16$$
$$x=-3 \qquad x=8$$

- **Square Root Property**
  - If $u^2 = d, then\ u = \sqrt{d}\ or\ u = -\sqrt{d}$
- **Solving Radicals Containing nth Roots**
  - 1) Arrange terms so one radical is isolated on one side of the equation.

  - 2) Raise both sides of the equation to the nth power to eliminate the nth root.
  - 3) Solve the resulting equation.
  - Check all solutions in the original equation.
    - Ex. $Solve: \sqrt{2x-1} + 2 = x$
    - Step 1: $\sqrt{2x-1} + 2 = x \rightarrow \sqrt{2x-1} = x - 2$
    - Step 2: $(\sqrt{2x-1})^2 = (x-2)^2 \rightarrow 2x - 1 = x^2 - 4x + 4$
    - Step 3: $2x - 1 = x^2 - 4x + 4$
    - $0 = x^2 - 6x + 5 \rightarrow 0 = (x-1)(x-5) \rightarrow x = 1 \text{ or } x = 5$

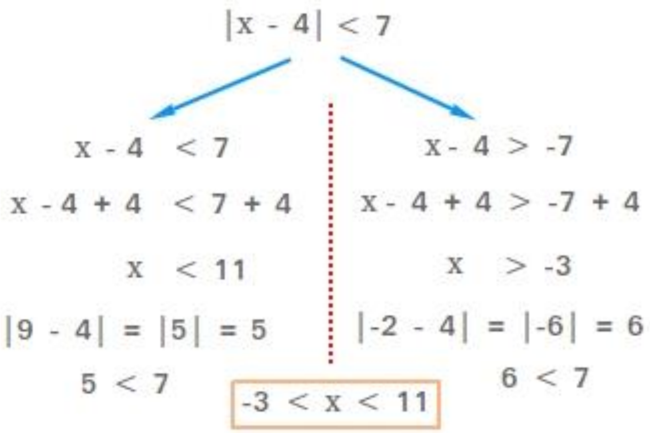
Linear Inequalities and Absolute Value Inequalities

- **Interval Notation:** represents subsets of real numbers
  - **Open Interval**: (a.b) represents the set of real numbers between, but not including a and b
  - **Closed Interval**: $[a, b]$represents the set of real numbers between, and including a and b
  - **Infinite Interval**: (a, ∞)represents the set of numbers that are greater than a
  - **Infinite Interval**: $(-\infty, b]$ represents the set of real numbers that are less than or equal to b
- **Graphing Intersections and Unions**
  - 1) Graph each **interval** on a number line
  - 2A) To find the **intersection**, find the set of numbers on the number line where both graphs have the set in common
  - 2B) To find the **union**, take the portion of the number line representing the total collection of numbers in the two graphs

Properties of Unions of Sets	
Commutative Property	$A \cup B = B \cup A$
Associative Property	$(A \cup B) \cup C = A \cup (B \cup C)$
Identity Property	$A \cup \varnothing = \varnothing \cup A$
Distributive Property	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **Solving an Absolute Value Inequality**

- If X is an algebraic expression and c is a positive number
    - 1) The solutions of $|X| < c$ are the numbers that satisfy $-c < X < c$
    - 2) The solutions of $|X| > c$ are the numbers that satisfy $X < -c$ or $X > c$



$|x - 4| < 7$

$x - 4 < 7$

$x - 4 + 4 < 7 + 4$

$x < 11$

$|9 - 4| = |5| = 5$

$5 < 7$

$x - 4 > -7$

$x - 4 + 4 > -7 + 4$

$x > -3$

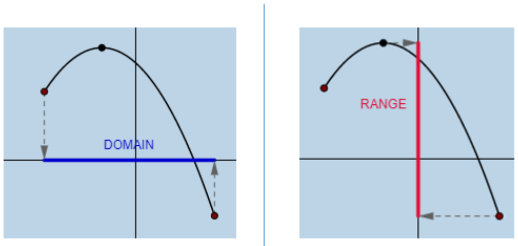
$|-2 - 4| = |-6| = 6$

$6 < 7$

$\boxed{-3 < x < 11}$

❖ Graphs and Functions

Graphs and Graphing Utilities



Domain is all the possible x values of a function.

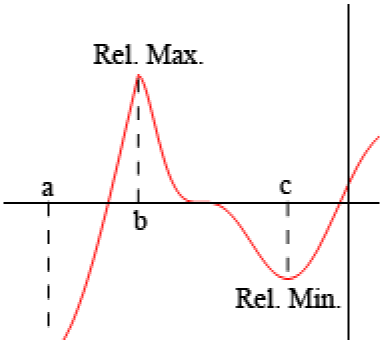
Range is all the possible y values of a function.

- **X- axis:** horizontal number line
- **Y- axis:** vertical number line
    - Each point corresponds to an ordered pair (x,y)

Basics of Functions and Their Graphs

- **Relation**: any set of ordered pairs
    - Set of first components is the **domain** (to find the domain, look for all inputs on the x-axis that correspond to the points on the graph). The set of second components is the **range** (to find the range, look for all the outputs on the y-axis that correspond to points on the graph
- **Vertical Line Test**: if any vertical line intersects the graph in more than one point, the graph is not a function
- **Zeros of a function**: values of x for which f(x) = 0
    - A function can have more than one x-intercept, but at most one y-intercept

- **Difference Quotient**: $\frac{f\ (x+h)-f(x)}{h}, h = 0$
- **Piecewise Function**: defined by two (or more) equations over a specified domain



- **Relative Maximum**: the "peak" of the graph
- **Relative Minimum:** the "bottom" of the graph

Function	**Even, Odd, or Neither?**
$f(x)=3x^2+8$	$f(-x)=3(-x)^2+8=3x^2+8=f(x)$ **Even!**
$f(x)=x^5-4x$	$f(-x)=(-x)^5-4(-x)=-x^5+4x$ $=-(x^5-4x)=-f(x)$ **Odd!**
$f(x)=2x^2-x-1$	$f(-x)=2(-x)^2-(-x)-1=2x^2+x-1$ $-f(x)=-(2x^2-x-1)=-2x^2+x+1$ $f(-x)\neq f(x)\neq -f(x)$ **Neither!**

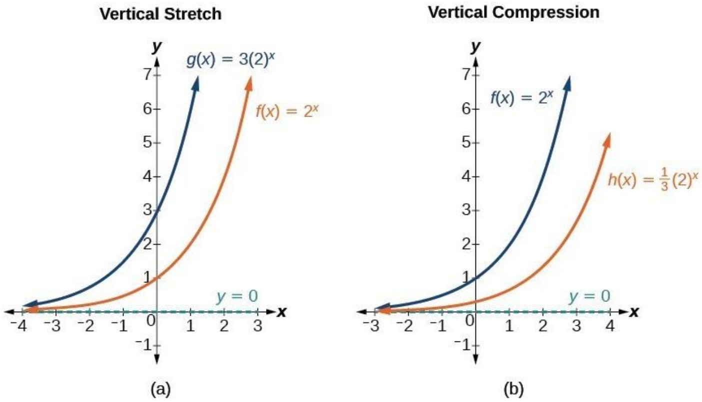
- **Even Function**: f(-x) = f(x); symmetric to the y-axis
- **Odd Function**: f(-x) = -f(x); symmetric to the origin

Linear Functions and Slope

- **To find slope:** $\frac{y^2-y^1}{x^2-x^1}$
- **Point Slope**: y-$y_1 = m(x - x_1)$
- **General Form**: Ax + By + C = 0
- **Average Rate of Change:** $\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x)}{x_2-x_1}$

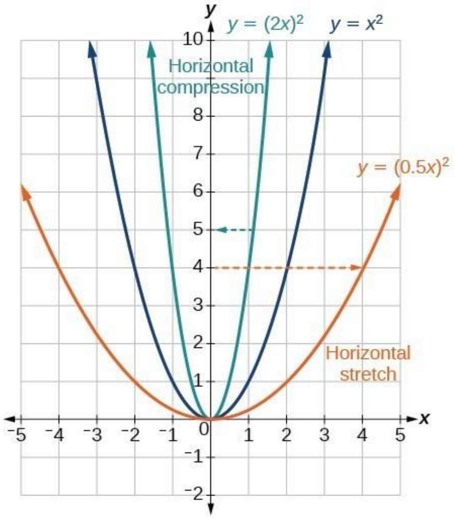
Transformations of Functions

- **Vertical Shifts**
  - The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted c units up.
  - The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted c units down.
- **Horizontal Shifts**
  - The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left c units.
  - The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted right c units.
- **Vertically Stretching and Shrinking Graphs**
  - In the graph $y = cf(x)$, if $c > 1$ the graph of $y = f(x)$ is vertically stretched by multiplying each of its y-coordinates by c.
  - In the graph $y = cf(x)$, if $0 < c < 1$, the graph $y = f(x)$ is vertically shrunk by multiplying each of its y-coordinates by c.



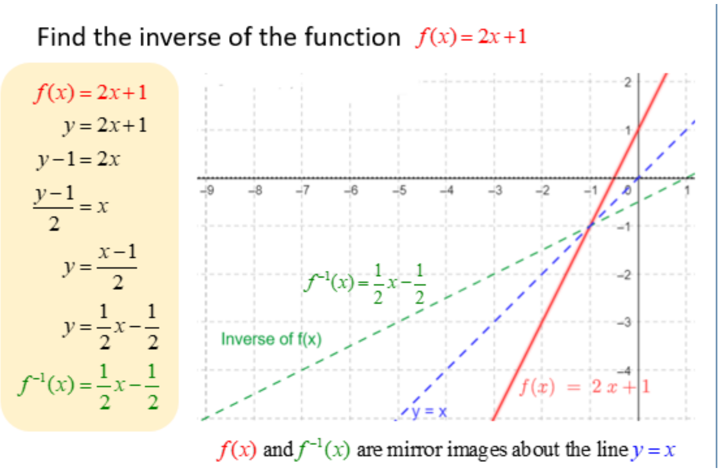
(a) (b)

- **Horizontally Stretching and Shrinking**
  - In the graph $y = f(cx)$, if $c > 1$ the graph of $y = f(x)$ is horizontally shrunk by dividing each of its x-coordinates by c.
  - In the graph of $y = f(cx)$, if $0 < c < 1$, the graph of $y = f(x)$ is horizontally stretched by diving each of its x-coordinates by c.



Inverse Functions

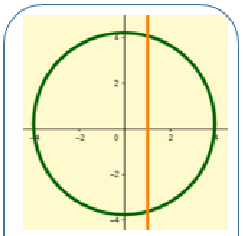
- **f(g(x)) = x and g(f(x)) = x,** the function g is the inverse of function f which means f and g are inverses of each other



Find the inverse of the function $f(x)=2x+1$

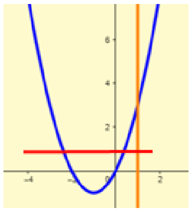
$$f(x)=2x+1$$
$$y=2x+1$$
$$y-1=2x$$
$$\frac{y-1}{2}=x$$
$$y=\frac{x-1}{2}$$
$$y=\frac{1}{2}x-\frac{1}{2}$$
$$f^{-1}(x)=\frac{1}{2}x-\frac{1}{2}$$

$f(x)$ and $f^{-1}(x)$ are mirror images about the line $y=x$



Fail Vertical Line Test

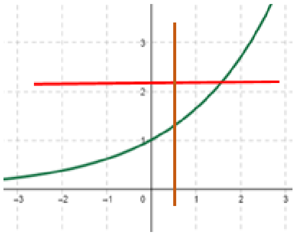
Not a Function



Pass Vertical Line Test

Fail Horizontal Line Test

Not a One-to-One Function



Pass Vertical Line Test

Pass Horizontal Line Test

A One-to-One Function

An invertible function
that has an inverse

- **Finding a Function's Inverse**
    - Switch x and y, then solve for y
- **Horizontal Line Test**: If function f has an inverse that is a function f^{-1}, if there is no horizontal line that intersects the graph of the function f at more than one point.

Distance and Midpoint Formulas: Circles

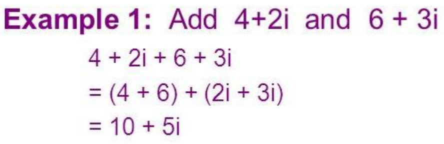
- **Distance Formula**: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
- **Midpoint Formula**: $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

- **Standard Form of Circle**: $(x-h)^2+(y-k)^2=r^2$
- **General Form of Circle Equation:** $x^2+y^2+Dx+Ey+F=0$

❖ Polynomial and Rational Functions

Complex Numbers

- **Imaginary Unit** $i$is defined as $i=\sqrt{-1}, where\ i^2=-1$
- In $a+bi, a$ is the real part and $b$is the imaginary part
- **Complex Numbers**: real numbers a and b, and i, the imaginary unit
- **Pure Imaginary Number:** imaginary number in the form of bi
- **Adding and Subtracting Complex Numbers**
  - $(a+bi)+(c+di)=(a+c)+(b+d)\,i$
    - Add complex numbers by adding the real parts, adding the imaginary parts, and expressing the sum as a complex number
  - $(a+bi)-(c+di)=(a-c)+(b-d)\,i$
    - Subtract complex numbers by subtracting the real parts, subtracting the imaginary parts, and expressing the sum as a complex number

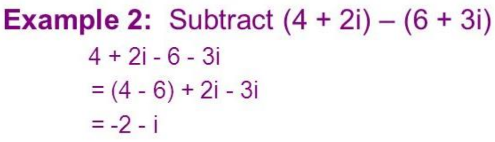


Example 1: Add 4+2i and 6 + 3i

4 + 2i + 6 + 3i

= (4 + 6) + (2i + 3i)

= 10 + 5i



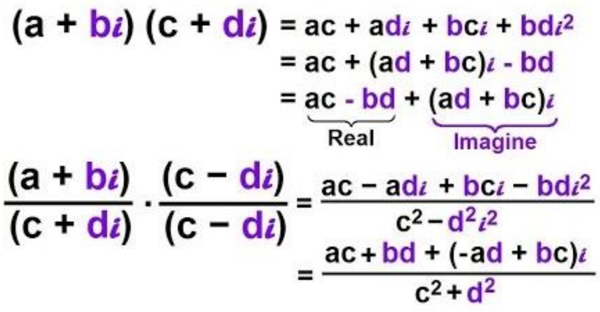
Example 2: Subtract (4 + 2i) – (6 + 3i)

4 + 2i - 6 - 3i

= (4 - 6) + 2i - 3i

= -2 - i

- **Multiplying and Dividing Complex Numbers**
  - For multiplication, FOIL, first, outer, inner, last, then solve.
  - For division, multiply the numerator and the denominator by the complex conjugate of the denominator. Use the FOIL method. Combine any imaginary terms, then combine real terms.



$$(a+bi)(c+di) = ac+adi+bci+bdi^2$$
$$= ac+(ad+bc)i-bd$$
$$= \underbrace{ac-bd}_{\text{Real}}+\underbrace{(ad+bc)i}_{\text{Imagine}}$$

$$\frac{(a+bi)}{(c+di)}\cdot\frac{(c-di)}{(c-di)}=\frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$$
$$=\frac{ac+bd+(-ad+bc)i}{c^2+d^2}$$

- **Complex Conjugate** of the number $a + bi$is $a - bi$
- **Principle Square Root**: for any positive real number b, the principal square root of the negative number $-b$is defined by $\sqrt{-b} = i\sqrt{b}$

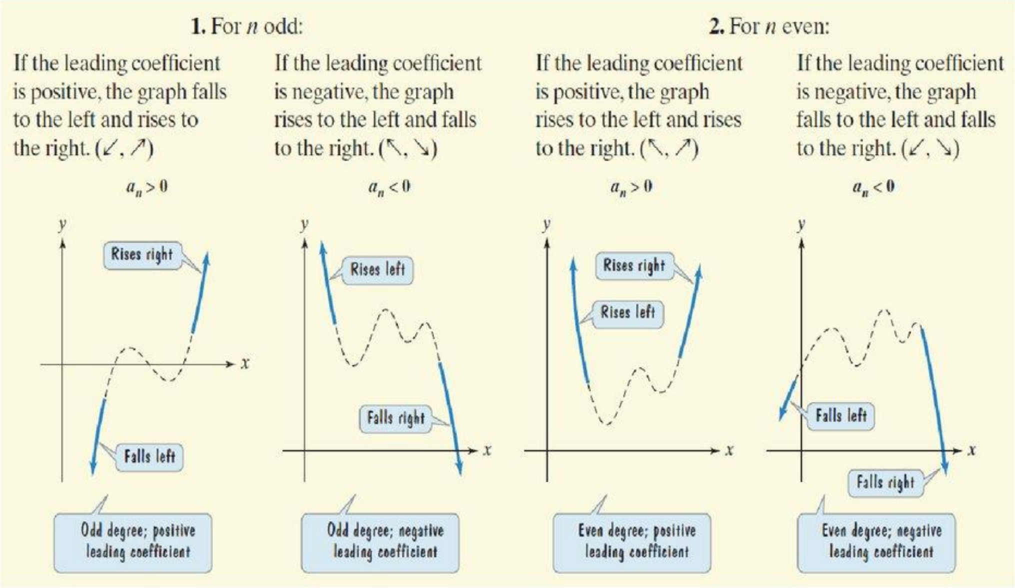
Quadratic Functions

- **Quadratic Function** is of the form: $f(x) = ax^2 + bx + c, a \neq 0$
- **Standard Form** of Quadratic Function: $f(x) = a(x-h)^2 + k, a \neq 0$
    - **Vertex**: (h,k)
- **Maximum and Minimum of Quadratic Functions**
    - Function $f(x) = ax^2 + bx + c$
        - If $a > 0$, then f has a minimum that occurs at $x = -\frac{b}{2a}$, minimum value $f(-\frac{b}{2a})$
        - If $a < 0$, then f has the maximum that occurs at $x = -\frac{b}{2a}$, maximum value $f(-\frac{b}{2a})$

Polynomial Functions and Their Graphs

- **Polynomial Function**: a function comprising more than one power function where the coefficients are assumed to not equal zero. The term with the highest degree is the leading term.

The Leading Coefficient Test for $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ *(continued)*



1. For *n* odd:

If the leading coefficient is positive, the graph falls to the left and rises to the right. (↙, ↗)

If the leading coefficient is negative, the graph rises to the left and falls to the right. (↖, ↘)

2. For *n* even:

If the leading coefficient is positive, the graph rises to the left and rises to the right. (↖, ↗)

If the leading coefficient is negative, the graph falls to the left and falls to the right. (↙, ↘)

- **Even multiplicity**: the graph touches the x-axis and turns around at r
- **Odd multiplicity**: the graph crosses the x-axis at r
- **Graphing a Polynomial Function**
  - 1) Use the **Leading Coefficient Test** to determine graph's end behavior
  - 2) Find **x-intercept** by setting f(x) = 0 and solving the resulting polynomial equation.

$P(x) = (x^3 + 3x^2)(- 4x - 12)$

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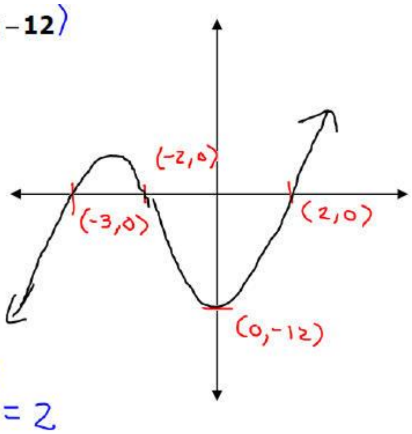
Section 4.1
End Behavior ↙↗
+ odd
yint x=0 (0,-12)
$x^2(x+3) - 4(x+3) = 0$
$(x+3)(x^2-4) = 0$
$x+3 \quad (x+2)(x-2) = 0$
$x = -3, \; x = -2, \; x = 2$



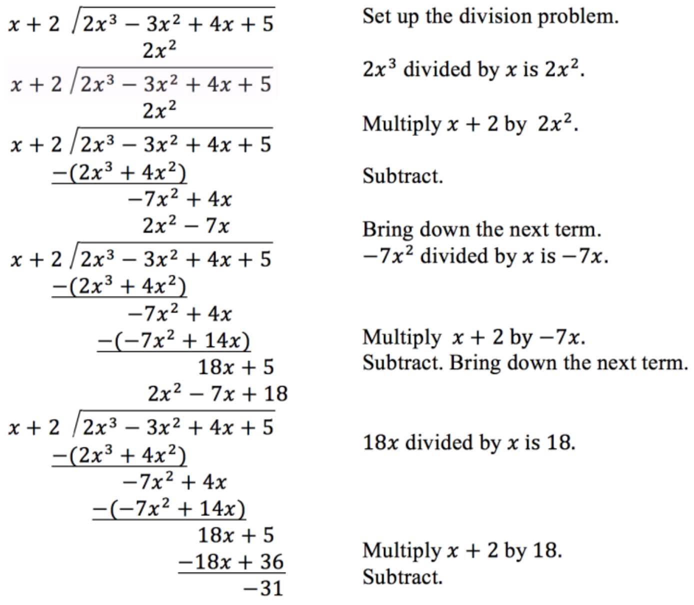
  - 3) Find the **y-intercept** by computing f(0)
  - 4) Use **symmetry**, if applicable to help draw the graph.
    - Y-axis symmetry: f(-x) = f(x)
    - Origin symmetry: f(-x) = -f(x)
  - 5) Use the fact that the maximum number of turning points of the graph is n-1 to

check whether its drawn correctly

- **Intermediate Value Theorem**: If f is a polynomial function and f(a) and f(b) have opposite signs, there is at least one value of c between a and b for which f(c) = 0.

Dividing Polynomials

- **Division Algorithm**
  - $f(x) = d(x) * q(x) + r(x)$
  - f(x) = dividend
  - d(x) = divisor
  - q(x) = quotient
  - r(x) = remainder
- **Long Division of Polynomials**
  - 1) Set up long division
  - 2) Divide the 1st term of the dividend with the divisor
  - 3) Multiply by the divisor
  - 4) Write the answer and subtract
  - 5) Bring down the next number to the right
  - 6) Repeat Step 2
  - 7) Write final answer



$x+2\,\overline{)\,2x^3-3x^2+4x+5}$

Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} \end{array}$$

$2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \end{array}$$

Multiply $x+2$ by $2x^2$.

Subtract.

$$\begin{array}{r} 2x^2-7x \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \end{array}$$

Bring down the next term.
$-7x^2$ divided by x is $-7x$.

Multiply $x+2$ by $-7x$.
Subtract. Bring down the next term.

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \\ \underline{-18x+36} \\ -31 \end{array}$$

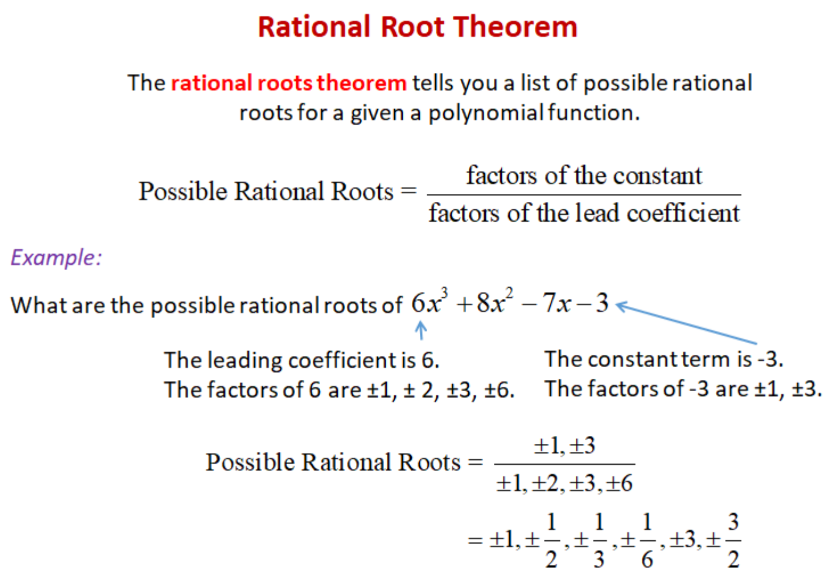
$18x$ divided by x is 18.

Multiply $x+2$ by 18.
Subtract.

- **Remainder Theorem**
  - If the polynomial f(x) is divided by x-c, then the remainder is f(c).
- **Factor Theorem**
  - Let f(x) be a polynomial
    - a) If f(c) = 0, then x-c is a factor of f(x)
    - b) If x-c is a factor of f(x), then f(c) = 0

Zeros of Polynomial Functions

- **Rational Zero Theorem**



Rational Root Theorem

The **rational roots theorem** tells you a list of possible rational roots for a given a polynomial function.

$$\text{Possible Rational Roots} = \frac{\text{factors of the constant}}{\text{factors of the lead coefficient}}$$

Example:

What are the possible rational roots of $6x^3+8x^2-7x-3$

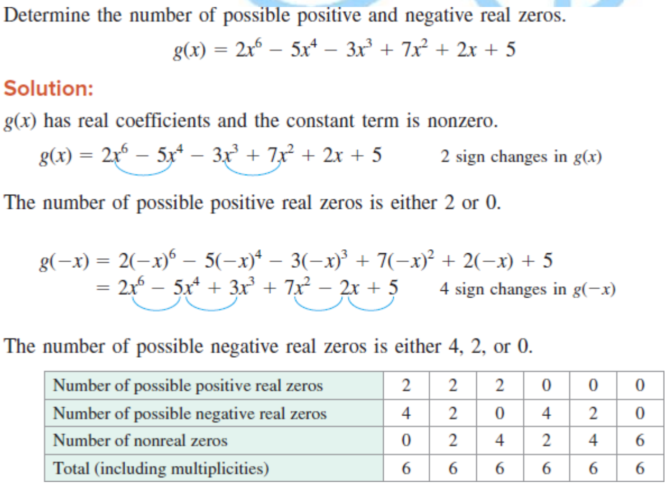
The leading coefficient is 6.
The factors of 6 are ±1, ± 2, ±3, ±6.

The constant term is -3.
The factors of -3 are ±1, ±3.

$$\text{Possible Rational Roots} = \frac{\pm1, \pm3}{\pm1, \pm2, \pm3, \pm6}$$

$$= \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm3, \pm\frac{3}{2}$$

- **Number of Roots**
  - If f(x) is a polynomial degree n ≥1, counting multiple roots separately, the equation f(x) = 0 has n roots.
- **Descartes's Rule of Signs**
  - The number of positive real zeros of f equals the number of sign changes of f(x) or is less than that number by an even integer. The number of negative real zeros of f applies a similar statement to f(-x).



Determine the number of possible positive and negative real zeros.

$$g(x) = 2x^6 - 5x^4 - 3x^3 + 7x^2 + 2x + 5$$

Solution:

$g(x)$ has real coefficients and the constant term is nonzero.

$$g(x) = 2x^6 - 5x^4 - 3x^3 + 7x^2 + 2x + 5 \qquad \text{2 sign changes in } g(x)$$

The number of possible positive real zeros is either 2 or 0.

$$g(-x) = 2(-x)^6 - 5(-x)^4 - 3(-x)^3 + 7(-x)^2 + 2(-x) + 5$$
$$= 2x^6 - 5x^4 + 3x^3 + 7x^2 - 2x + 5 \qquad \text{4 sign changes in } g(-x)$$

The number of possible negative real zeros is either 4, 2, or 0.

Number of possible positive real zeros	2	2	2	0	0	0
Number of possible negative real zeros	4	2	0	4	2	0
Number of nonreal zeros	0	2	4	2	4	6
Total (including multiplicities)	6	6	6	6	6	6

Rational Functions and Their Graphs

- **Arrow Notation**
  - $x \to a^+$: x approaches a from the right

  - $x \to a^-$: x approaches a from the left
  - $x \to \infty$: x approaches infinity
  - $x \to -\infty$: x approaches negative infinity
- **Vertical Asymptote**
  - If f(x) increases or decreases without bond as x approaches a
- **Horizontal Asymptote**
  - If f(x) approaches b as x increases or decreases without bond
- **Graphing Rational Functions** $f(x) = \frac{p(x)}{q(x)}$
  - Determine whether the graph has **symmetry**
    - f(-x) = f(x): y-axis symmetry
    - f(-x) = -f(x): origin symmetry
  - Find the **y-intercept** by evaluating f(0)
  - Find the **x-intercept** by solving the equation p(x)=0
  - Find any **vertical asymptotes** by solving the equation q(x) = 0
  - Find the **horizontal asymptote** by using the rule for determining the horizontal asymptote of a rational function.
  - Plot at least one point between and beyond each x-intercept and vertical asymptote.
  - Use info above to graph the function between the asymptotes

❖ Exponential and Logarithmic Functions

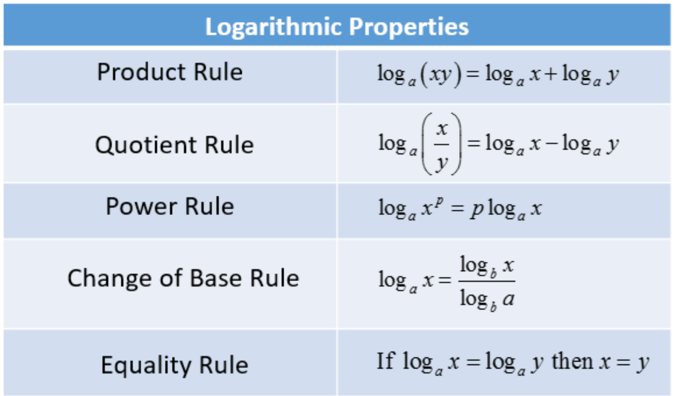
Exponential Functions

- **Parent Function:** $f(x) = b^x$, where b is base and b > 0, $b \neq 1$
- **Natural exponential function:** $f(x) = e^x$

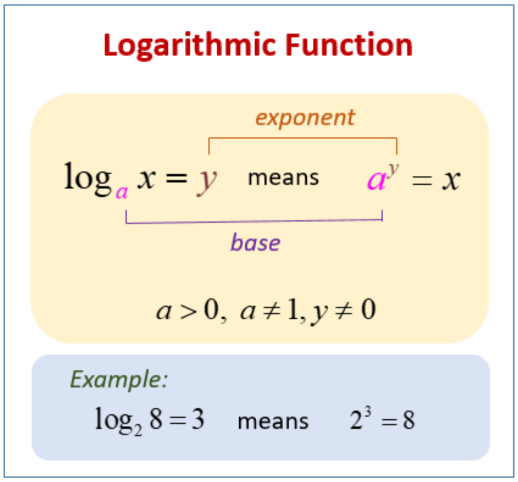
- **Irrational number (natural base):** $e \approx 2.7183$, $e$is the value that $(1+\frac{1}{n})^2$

Logarithmic Functions

- **Logarithmic Functions**: for x>0 and b>0, b≠ *1*



Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$



Logarithmic Function

$$\underbrace{\log_a x = \overbrace{y \quad \text{means} \quad a^y}^{\text{exponent}}}_{\text{base}} = x$$

$$a > 0,\ a \neq 1, y \neq 0$$

Example:

$\log_2 8 = 3$ means $2^3 = 8$

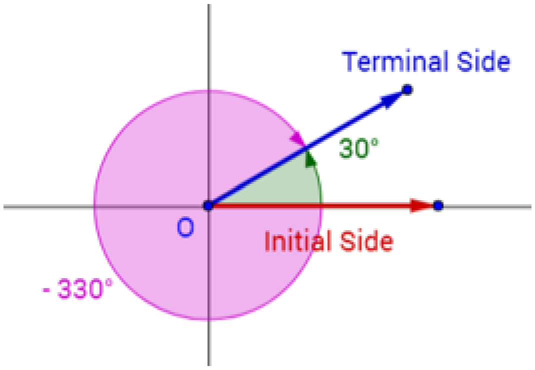
Exponential and Logarithmic Equations

- **Exponential equation**: equation containing a variable in an exponent
- **Solving Exponential Equations by Expressing Each Side as a Power of the Same Base**
  - If $b^M = b^N, then\ M = N$
- **Using Natural Logarithms to Solve Exponential Equations**
  - 1) Isolate the exponential expression
  - 2) Take the natural logarithm on both sides of the equation
  - 3) Simplify using one of the following
    - $In\ (b^x) = x\ ln\ (b) \quad or \quad ln\ e^x = x$
  - 4) Solve for the variable
- **Using the Definition of a Logarithm to Solve Logarithmic Equations**
  - 1) Express the equation in the form $log_b M = c$.
  - 2) Use the definition to rewrite the equation in exponential form $log_b M = c\ means\ b^c = m$
  - 3) Solve for the variable.

❖ Trigonometric Functions

Angles and Radian Measure

- **Angle**: two rays with a common endpoint called the vertex
- **Quadrantal Angle:** angle with its terminal side on the x-axis or the y-axis
- **Radian Measure:** $\theta = \frac{s}{r} radians$
- To **convert from degrees to radians**, multiply by $\frac{\pi radians}{180\circ}$
- To **convert from radians to degrees**, multiply by $\frac{180\circ}{\pi radians}$

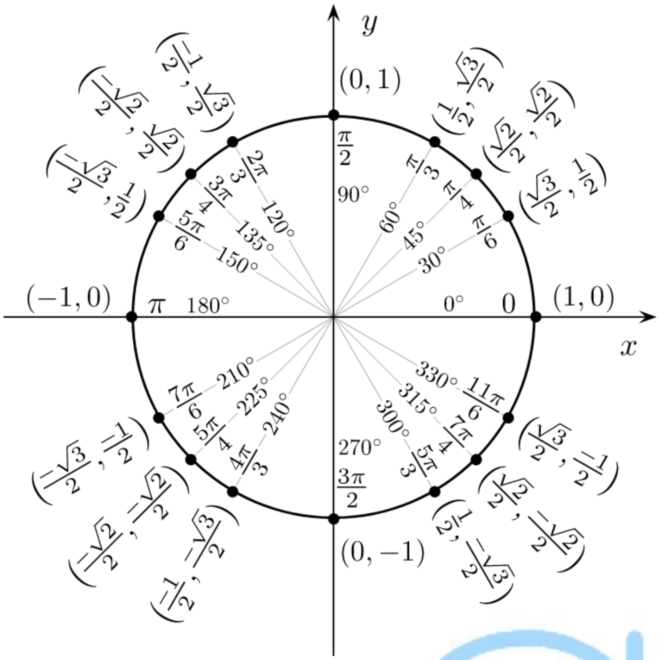


- **Coterminal Angles**: angles with the same initial and terminal sides
- **Length of a Circular Arc**: $s = r\theta$

The Unit Circle

- **Definitions of the Trigonometric Functions in Terms of a Unit Circle**
    - If t is a real number and $P = (x, y)$ is a point on the unit circle that corresponds to t, then

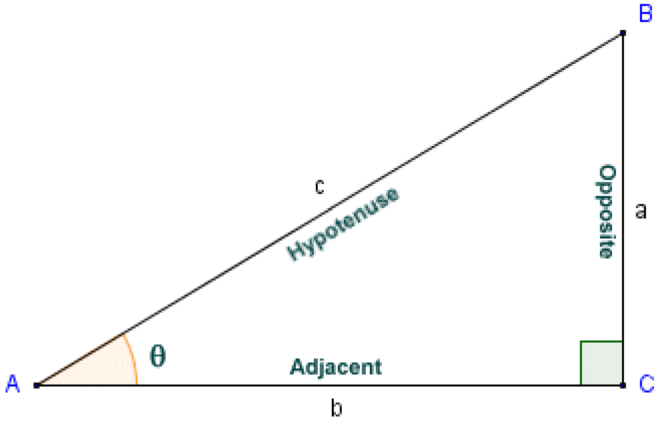
$sin\, t = y$	$cos\, t = x$	$tan\, t = \frac{y}{x}, x \neq 0$
$csc\, t = \frac{1}{y}, y \neq 0$	$sec\, t = \frac{1}{x}, x \neq 0$	$cot\, t = \frac{x}{y}, y \neq 0$



- **Domain and Range of Sine and Cosine Functions**
  - **Domain** to sine and cosine function $(-\infty, \infty)$, set of all real numbers
  - **Range** $[-1,1]$
- **Even and Odd Trigonometric Functions**
  - **Even**
    - $cos\ (-t) = cos\ t\ and\ sec(-t) = sec\ t$
  - **Odd**
    - $sin(-t) = \ -sin\ t, tan(-t) = \ -tan(t), csc(-t) = -csc\ t, cot(-t) = -\ cot(t)$
- **Quotient Identities**
  - $tan\ t = \frac{sin\ t}{cos\ t}\ and\ cot\ t\ = \frac{cos\ t}{sin\ t}$
- **Pythagorean Identities**
  - $sin^2 + cos^2 t = 1 \qquad 1 + tan^2 t = sec^2 t \qquad 1 + cot^2 t = csc^2 t$
- **Periodic Functions**: a function that repeats its values at regular intervals
  - **Periodic Properties for Sine and Cosine (period 2**π**)**
    - $sin\ (t + 2\pi) = sin\ t\ and\ cos\ (t + 2\pi) = cos\ t$
  - **Periodic Properties for Tangent and Cotangent (period** π**)**
    - $tan\ (t + \pi) = tan\ t\ and\ cot\ (t + \pi) = cot\ t$

Right Triangle Trig

- **Right Triangle Definitions for Trig Functions**



$\sin\theta = \frac{a}{c}$	$\cos\theta = \frac{b}{c}$	$\tan\theta = \frac{a}{b}$
$\csc\theta = \frac{c}{a}$	$\sec\theta = \frac{c}{b}$	$\cot\theta = \frac{b}{a}$

- **Cofunction Identities**

$\sin\theta = \cos(90° - \theta)$	$\cos\theta = \sin(90° - \theta)$	$\tan\theta = \cot(90° - \theta)$
$\cot\theta = \tan(90° - \theta)$	$\sec\theta = \csc(90° - \theta)$	$\csc\theta = \sec(90° - \theta)$

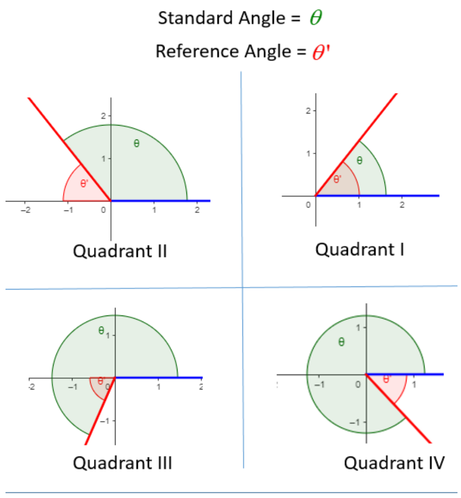
Trigonometric Functions of Any Angle

- If $r = \sqrt{x^2 + y^2}$ is the distance from (0,0) to (x,y), the six trig functions of θ are

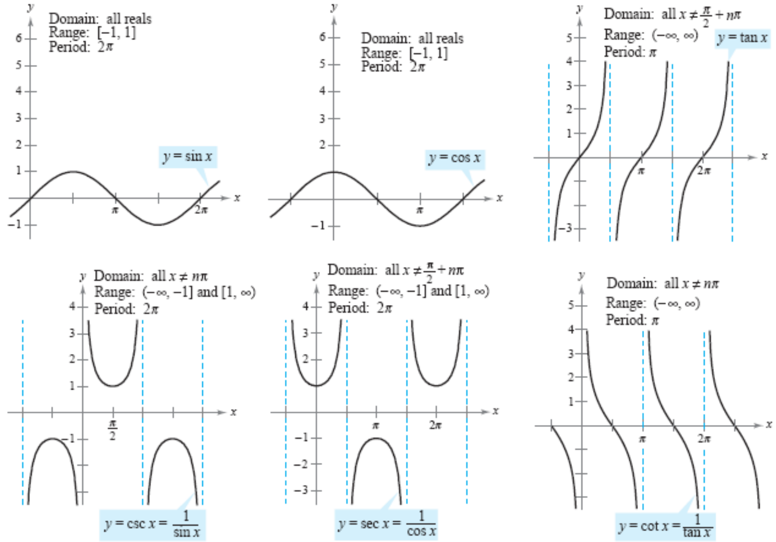
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}, y \neq 0$
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}. x \neq 0$
$\tan\theta = \frac{y}{x}, x \neq 0$	$\cot\theta = \frac{x}{y}, y \neq 0$

From https://simplestudies.edublogs.org

- **Reference Angle**: positive acute angle between the terminal side and x-axis
- **Find Reference Angles for Angles Greater Than 360°(2π) or Less Than -360° (-2π)**
  - Find a positive angleα less than 360° that is coterminal with the given angle.
  - Drawα in standard position.
  - Use the drawing to find the reference angle for the given angle.



Graphs of Trig Functions

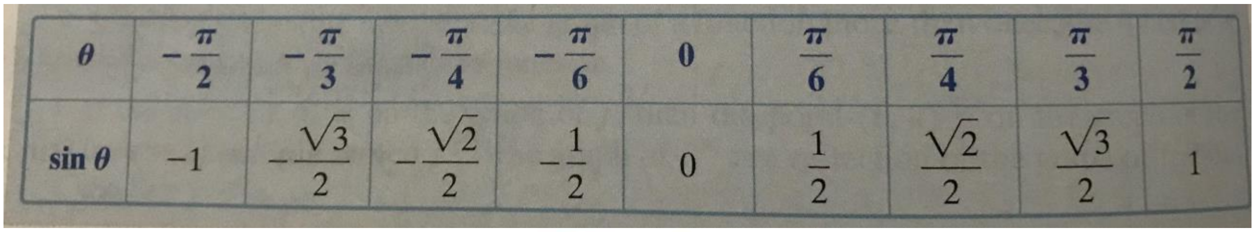


The graphs of the six trigonometric functions

- The graph of $y = A\ sin(Bx + C)$can be obtained using **amplitude** $|A|$, **period** $\frac{2\pi}{b}$, and **phase shift** $\frac{C}{B}$.
- The graph of $y = A\ cos(Bx - C)$ can be obtained using **amplitude** $|A|$, **period** $\frac{2\pi}{b}$, and **phase shift** $\frac{C}{B}$.
- $y = A\ sin(Bx + C) + D\ and\ y = A\ cos(Bx - C) + D$, the constant D can cause **vertical shifts**. If $D > 0$, you shift **upward**. If $D < 0$, you shift **downward**.

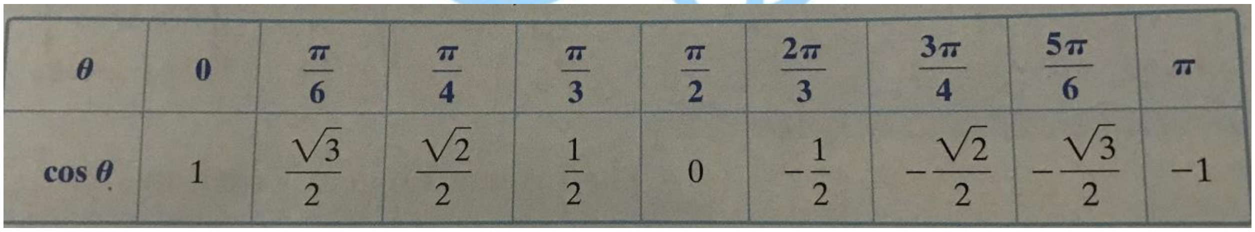
Inverse Trig Functions

- **Inverse Sine Function**: inverse of the restricted sine function $y = sin\ x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$which means $y = sin^{-1}x\ means\ sin\ y = x$
- **Finding Exact Values of** $sin^{-1}x$
    - 1) Let $\theta = sin^{-1}x$
    - 2) Rewrite $\theta = sin^{-1}x$ as $sin\theta = x, where\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
    - 3) Use the table to find the exact values that satisfies $sin\theta = x$



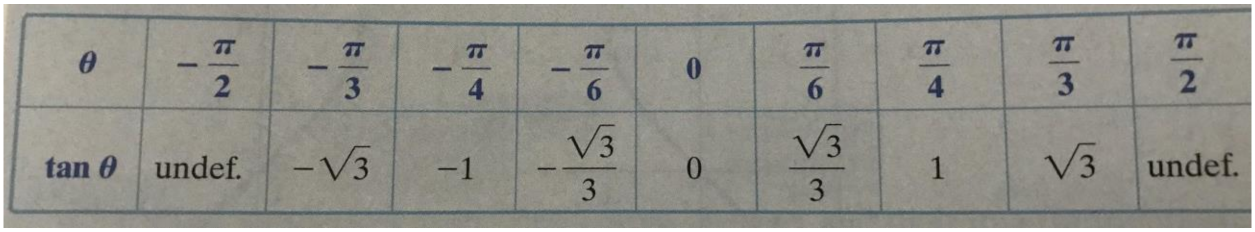
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

- **Inverse Cosine Function**: inverse of the restricted cosine function $y = cos\ x, 0 \leq x \leq \pi$which means $y = cos^{-1}x\ means\ cos\ y = x$
- **Finding Exact Values of** $cos^{-1}x$
    - 1) Let $\theta = cos^{-1}x$
    - 2) Rewrite $\theta = cos^{-1}x$as $cos\theta = x$, where $0 \leq \theta \leq \pi$
    - 3) Use the table to find the exact values



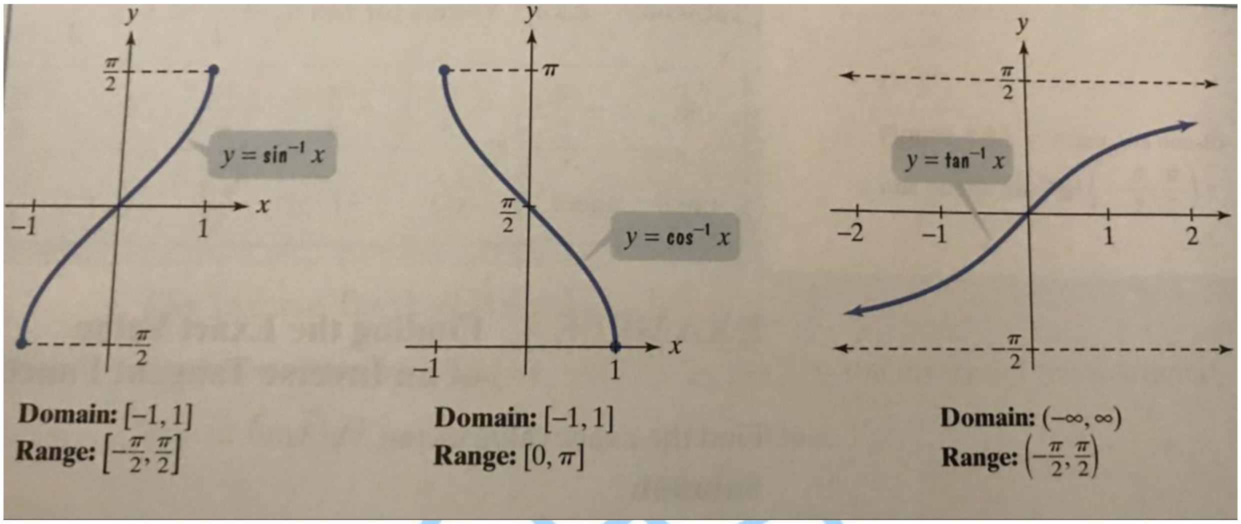
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

- **Inverse Tangent Function:** inverse of the restricted tangent function $y = cos\ x, 0 \leq x \leq \pi$which means $y = tan\ x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- **Finding Exact Values of** $tan^{-1}x$
    - 1) Let $\theta = tan^{-1}x$
    - 2) Rewrite $\theta = tan^{-1}x\ as\ tan\ \theta = x\ , where\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
    - 3) Use the table to find exact values



θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan\theta$	undef.	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

- **Graphs of Three Basic Inverse Trig Functions**



Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain: $[-1, 1]$
Range: $[0, \pi]$

Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

❖ Analytic Trigonometry

Verifying Trig Identities

- **Fundamental Trig Identities**
  - **Reciprocal Identities**

$\sin x = \frac{1}{\csc x}$	$\cos x = \frac{1}{\sec x}$	$\tan x = \frac{1}{\cot x}$
$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$

  - **Quotient Identities**

$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$

  - **Pythagorean Identities**

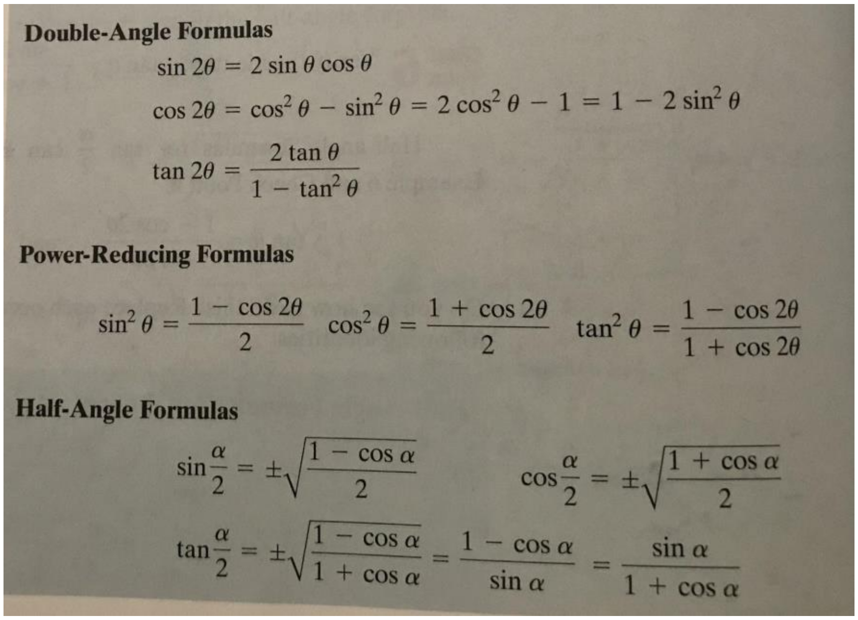
$\sin^2 x + \cos^2 x = 1$	$1 + \tan^2 x = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$

  - **Even-Odd Functions**

$sin\ (-x) = -sin\ x$	$cos\ (-x) = cos\ x$	$tan\ (-x) = \ -tan\ x$
$csc\ (-x) = \ -csc\ x$	$sec\ (-x)\ = sec\ x$	$cot\ (-x) = -cot\ x$

- **Principal Trig Identities**

$sin\ (\alpha+\beta) = sin\ \alpha\ cos\ \beta + cos\ \alpha\ sin\ \beta$	$sin\ (\alpha-\beta) = sin\ \alpha\ cos\ \beta -\ cos\ \alpha\ sin\ \beta$
$cos\ (\alpha+\beta) = cos\ \alpha\ cos\beta - sin\ \alpha\ sin\ \beta$	$cos\ (\alpha-\beta) = cos\ \alpha\ cos\ \beta + sin\ \alpha\ sin\ \beta$
$tan\ (\alpha+\beta) = \frac{tan\ \alpha + tan\ \beta}{1 - tan\ \alpha\ tan\beta}$	$tan\ (\alpha-\beta) = \frac{tan\ \alpha - tan\ \beta}{1 + tan\ \alpha\ tan\beta}$



Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

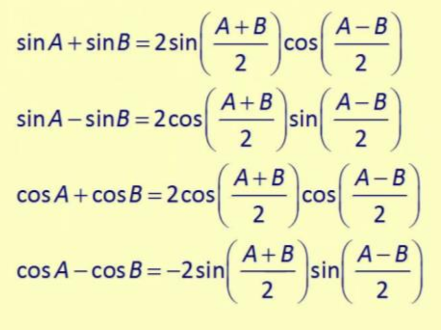
Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

- ○ Product To Sum Formula

$sin\ \alpha\ sin\ \beta = \frac{1}{2}[cos\ (\alpha-\beta) - cos(\alpha+\beta)]$
$cos\ \alpha\ cos\ \beta = \frac{1}{2}[cos\ (\alpha-\beta)\ +\ cos(\alpha+\beta)]$
$sin\ \alpha\ cos\ \beta = \frac{1}{2}[sin\ (\alpha+\beta) - sin(\alpha-\beta)]$
$cos\ \alpha\ sin\ \beta = \frac{1}{2}[sin\ (\alpha+\beta) - sin(\alpha-\beta)]$

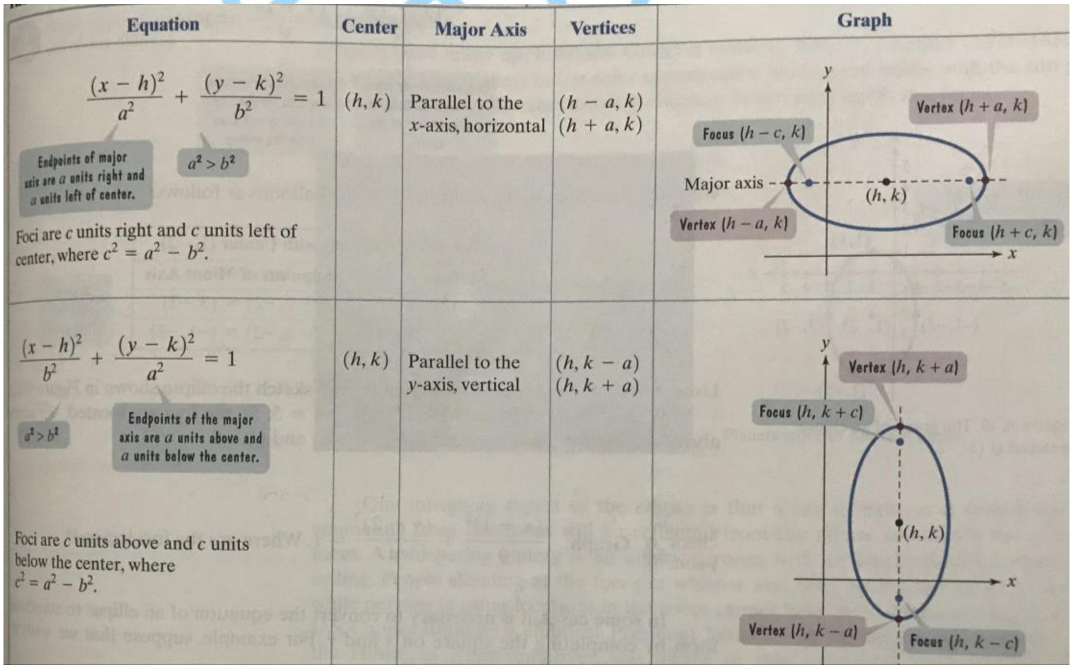


$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

→ *Sum to Product Formulas*

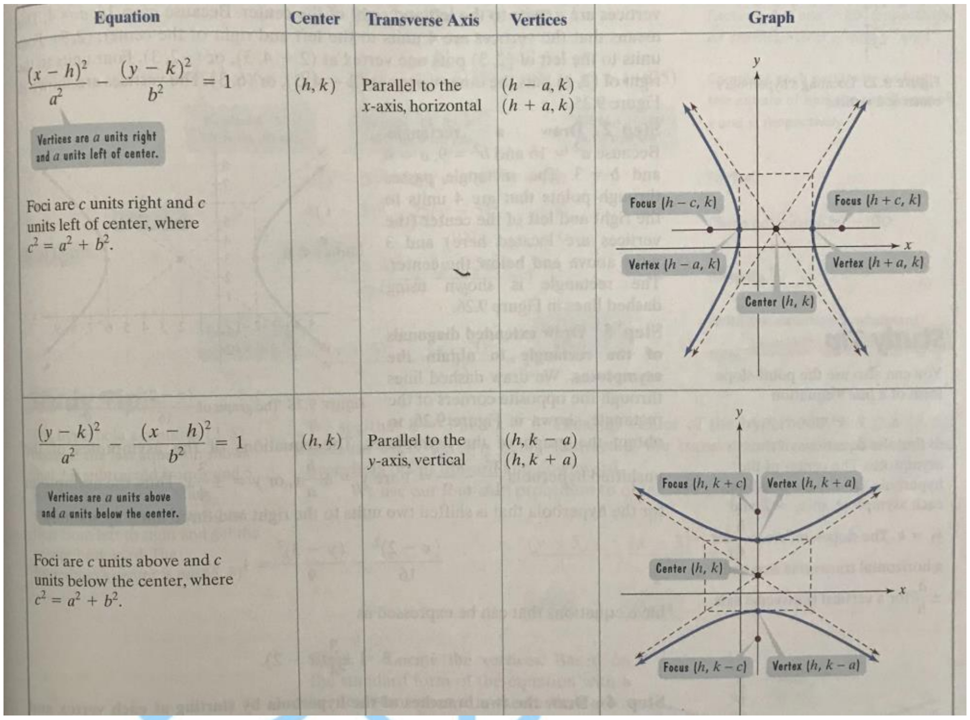
❖ Conic Sections

- **The Ellipse:** is the set of all points P in a plane the sum of whose distances from two fixed points is constant
  - ○ **Foci**: the two fixed points
  - ○ **Center**: midpoint of the segment connecting the foci
- **Standard Form Of Ellipse**
  - ○ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *or* $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
- **Standard Form of Equations for Ellipses**
  - ○



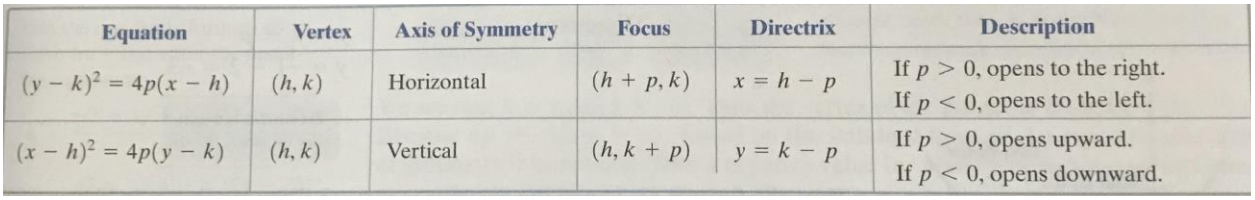
Equation	Center	Major Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ Endpoints of major axis are a units right and a units left of center. $a^2>b^2$ Foci are c units right and c units left of center, where $c^2=a^2-b^2$.	(h, k)	Parallel to the x-axis, horizontal	$(h-a, k)$ $(h+a, k)$	
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $a^2>b^2$ Endpoints of the major axis are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2=a^2-b^2$.	(h, k)	Parallel to the y-axis, vertical	$(h, k-a)$ $(h, k+a)$	

- **The Hyperbola:** a set of points in a plane the difference of whose distances from two fixed points, called foci is constant

  - **Vertices**: line through the foci that intersects at two points
  - **Standard Forms of the Equations of a Hyperbola**
    - $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$
  - **Standard Forms of Hyperbolas Centered at (h.k)**
    - 

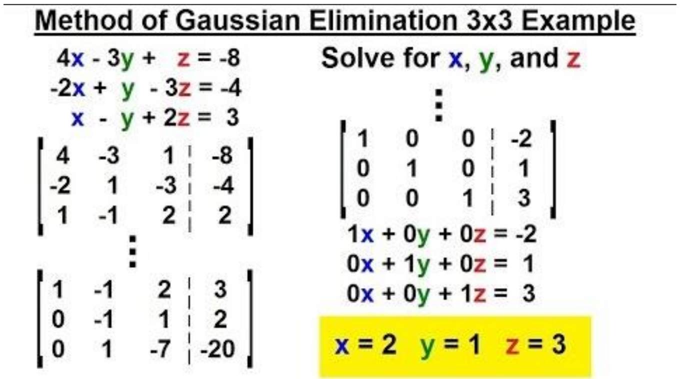
Equation	Center	Transverse Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Vertices are a units right and a units left of center. Foci are c units right and c units left of center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the x-axis, horizontal	$(h - a, k)$ $(h + a, k)$	
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Vertices are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the y-axis, vertical	$(h, k - a)$ $(h, k + a)$	

- **The Parabola**: set of all points in a plane that are equidistant from a fixed line.
  - **Directrix**: a fixed point
  - **Focus**: not on the line
  - **Latus Rectum**: line segment that passes through its focus, parallel to the directrix, the endpoints are located on the parabola
    - Latus rectum is $|4p|$
- **Standard Form of the Equation of a Parabola**
  - $y^2 = 4px \text{ or } x^2 = 4py$



Equation	Vertex	Axis of Symmetry	Focus	Directrix	Description
$(y - k)^2 = 4p(x - h)$	(h, k)	Horizontal	$(h + p, k)$	$x = h - p$	If $p > 0$, opens to the right. If $p < 0$, opens to the left.
$(x - h)^2 = 4p(y - k)$	(h, k)	Vertical	$(h, k + p)$	$y = k - p$	If $p > 0$, opens upward. If $p < 0$, opens downward.

❖ Matrices and Determinants



Method of Gaussian Elimination 3x3 Example

$$\begin{aligned} 4x - 3y + z &= -8 \\ -2x + y - 3z &= -4 \\ x - y + 2z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & -8 \\ -2 & 1 & -3 & -4 \\ 1 & -1 & 2 & 2 \end{array}\right]$$

⋮

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & -7 & -20 \end{array}\right]$$

Solve for x, y, and z

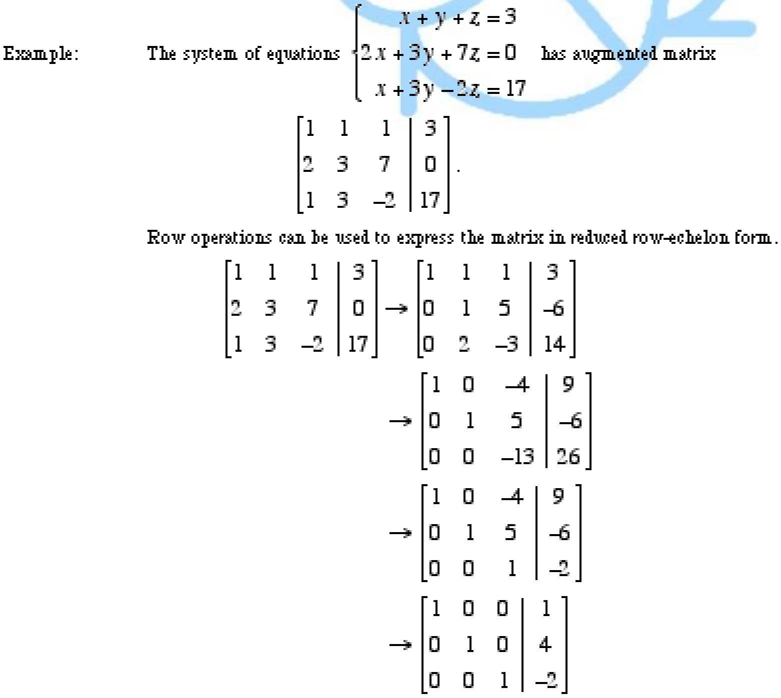
⋮

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} 1x + 0y + 0z &= -2 \\ 0x + 1y + 0z &= 1 \\ 0x + 0y + 1z &= 3 \end{aligned}$$

x = 2 y = 1 z = 3

- **Augmented Matrices:** has a vertical bar separating the columns of the matrix into 2 groups
- **Row-Echelon Form**: matrix with 1s down the main diagonal and 0s below the 1s
- **Gaussian Elimination**: process used to solve linear systems using matrix row operations
- **Gauss-Jordan Elimination**: reduced row-echelon form, the process for a matrix with 1s down the main diagonal and 0s in every position above and below each 1 is found



Example: The system of equations $\begin{cases} x + y + z = 3 \\ 2x + 3y + 7z = 0 \\ x + 3y - 2z = 17 \end{cases}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

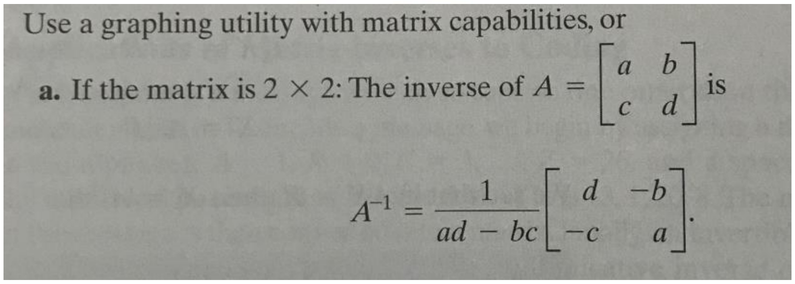
$$\to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

- **Matrix Addition and Subtraction:** matrices of the same order are added or subtracted by adding or subtracting.
    - **Properties of Matrix Addition**
        - 1) A+B = B+A
        - 2) (A+B) + C = A + (B+C)
        - 3) A+0 = 0+A =A
        - 4) A + (-A) = (-A) + A = 0
- **Scalar Multiplication**: product of a real number and a matrix

$$2 \cdot \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 & 2 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 3 \end{bmatrix}$$

    - **Properties of Scalar Multiplication**
        - (cd)A = c(dA)
        - 1A=A
        - c(A+B) = cA+cB
        - (c+d)A = cA+dA
    - **Properties of Matrix Multiplication**
        - (AB)C= A(BC)
        - A(B+C)= AB+ AC
        - (A+B) C = AC + BC
        - c(AB) = (cA) B
- **Finding Multiplicative Inverses for Invertible Matrices**

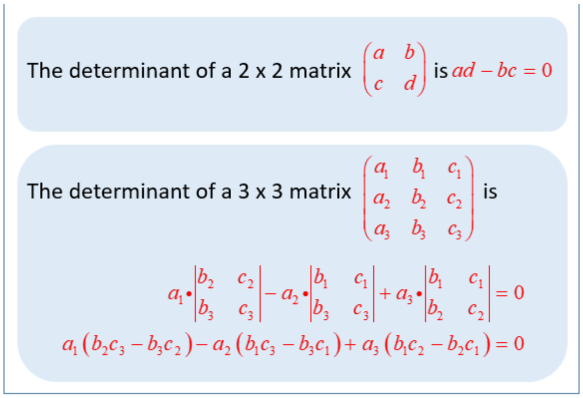


Use a graphing utility with matrix capabilities, or

a. If the matrix is 2 × 2: The inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- **Finding the Determinant of a 2x2 Matrix**



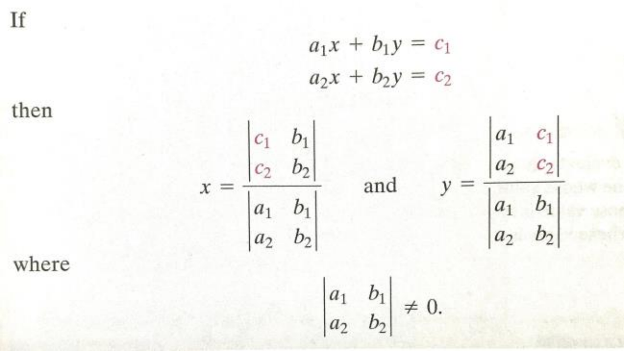
The determinant of a 2 x 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc = 0$

The determinant of a 3 x 3 matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is

$$a_1 \bullet \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \bullet \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \bullet \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0$$

$$a_1\left(b_2c_3 - b_3c_2\right) - a_2\left(b_1c_3 - b_3c_1\right) + a_3\left(b_1c_2 - b_2c_1\right) = 0$$

- **Cramer's Rule:** method of using determinants to solve the linear equation



If

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

then

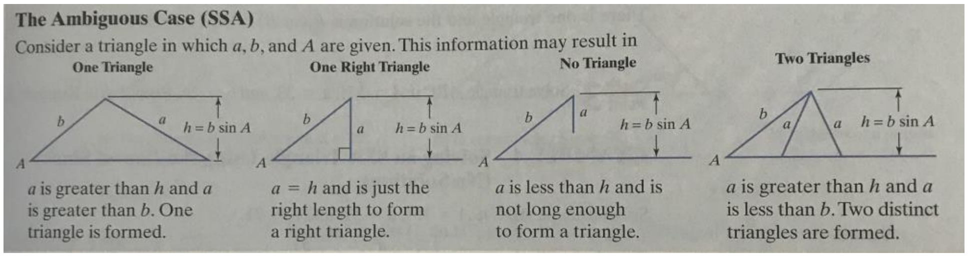
$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

❖ Additional Topics in Trigonometry

- **Law of Sines**
  - $\dfrac{a}{sin\ A} = \dfrac{b}{sin\ B} = \dfrac{c}{sin\ C}$



The Ambiguous Case (SSA)

Consider a triangle in which a, b, and A are given. This information may result in

a is greater than h and a is greater than b. One triangle is formed.

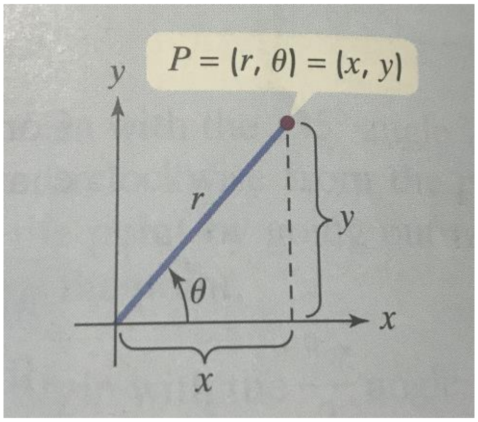
$a = h$ and is just the right length to form a right triangle.

a is less than h and is not long enough to form a triangle.

a is greater than h and a is less than b. Two distinct triangles are formed.

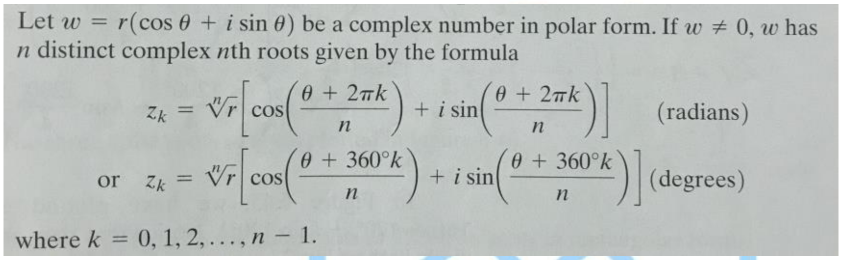
  - **Ambiguous Case:** given information may result in one triangle, two triangles, or no triangle at all
- **Law of Cosine**
  - $a^2 = b^2 + c^2 - 2bc\ cos\ A$
  - $b^2 = a^2 + c^2 - 2ac\ cos\ B$
  - $c^2 = a^2 + b^2 - 2ab\ cos\ C$

- **Polar Coordinates:** (r, θ)
- **Rectangular Coordinates**: (x,y)
- **Multiple Representations of Points in the Coordinate System**
    - If $n$is any integer, the point (r, θ)can be represented as $(r, \theta) \ = (r, \theta + 2n\pi) \ or \ (r, \theta) = (-r, \theta + \pi + 2n\pi)$
- **Relations between Polar and Rectangular Coordinates**



- $x = r \cos \theta$
- $y = r \sin \theta$
- $x^2 + y^2 = r^2$
- $tan\theta = \frac{y}{x}$
- **Converting a Point from Rect. to Polar Coordinates** $(r > 0 \ and \ 0 \leq \theta \leq 2\pi$
    - 1) Plot the point (x,y)
    - 2) Find r by computing the distance from the origin to (x,y): $r = \sqrt{x^2 + y^2}$
    - 3) Find θusing tan $\theta = \frac{y}{x}$with the terminal side passing through (x,y)
- **Absolute Value of a Complex Number**
    - Absolute value $a + bi$ is $|z| = |a + bi| = \sqrt{a^2 + b^2}$
- **Polar Form of a Complex Number**
    - The complex number $z = a + bi$is written in polar form as $z = r(\cos \theta + i \sin \theta)$, where $a = r \cos \theta, b = r \sin \theta, r = \sqrt{a^2 + b^2}$,and $\tan \theta = \frac{b}{a}$.
        - **Modulus**: value of r
        - **Argument**: value of θ
- **Product of Two Complex Numbers in Polar Forms**

- Let $z_1 = cos\theta_1 + i\ sin\theta_1)\ and\ z_2 = cos\theta_2 + i\ sin\theta_2)$, the product would be $z_1 z_2 = r_1 r_2[cos(\theta_1 + \theta_2) + i\ sin(\theta_1 + \theta_2)]$.
    - To multiply 2 complex #s, multiply the moduli and add arguments.
- **Quotient of Two Complex Numbers in Polar Form**
  - Let $z_1 = r_1(cos\theta_1 + i\ sin\theta_1)\ and\ z_2 = r_2(cos\theta_2 + i\ sin\theta_2)$, the quotient would be $\frac{z_1}{z_2}$.
- **DeMoivre's Theorem:** formula for the nth power and multiplying the argument by n
  - $Let\ z = r(cos\theta + i\ sin\theta)$ be a complex number in polar form. If $n$is a positive integer, then $z$to the nth power, $z^n$is $z^n = [r(cos\theta\ \ + i\ sin\theta)]^n = r^n(cos\ n\theta + i\ sin\ n\theta)$.
- **Using DeMoivre's Theorem for Finding Complex Roots**



Let $w = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, w has n distinct complex nth roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{(radians)}$$

or $$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right] \quad \text{(degrees)}$$

where $k = 0, 1, 2, \ldots, n - 1$.

- **Vectors**: quantities that involve both a magnitude and a direction (usually denoted with v)
- **Scalars**: quantities that involve magnitude, but no direction
- **i and j Unit Vectors**
  - i - direction is along the positive x-axis
  - j - direction is along the positive y-axis
- **Adding and Subtracting Vectors in Terms of i and j**
  - If $a_1 i + b_1\ j\ and\ w = a_2 i\ + b_2\ j, then\ v + w = (a_1 + a_2)i + (b_1 + b_2)j\ \ or\ v - w = (a_1 - a_2)i + (b_1 - b_2)j$
- **Scalar (k) Multiplication with a Vector in Terms of i and j**
  - $kv = (ka)i + (kb)\ j$
- **Properties of Vector Addition and Scalar Multiplication**
  - **Vector Addition Properties**
    - $u + v = v + u$
    - $(u + v) + w = u + (v + w)$

        - $u + 0 = 0 + u = u$
        - $u + (-u) = (-u) + u = 0$
    - **Scalar Multiplication Properties**
        - $(cd)u = c(du)$
        - $c(u + v) = cu + cv$
        - $(c + d)u = cu + du$
        - $1u = u$
        - $0u = u$
        - $||cv|| = |c|||v||$
- **The Dot Product**: is defined as $v \cdot w = a_1 a_2 + b_1 b_2$
- **Properties of the Dot Product**
    - If $u, v, and\ w$are vectors, and c is a scalar then,
        - $u \cdot v = v \cdot u$
        - $u \cdot (v + w) = u \cdot v + u \cdot w$
        - $0 \cdot v = 0$
        - $v \cdot v = ||v||^{2}$
        - $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
- **Alt. Formula for The Dot Product**
    - $v \cdot w = ||v||||w||cos\ \theta$
- **Vector Projection of v Onto w**
    - $proj_w v = \frac{v \cdot w}{||w||^{2}} w$

******NOTE: We do not claim ownership of any images used in this study guide. Some definitions, examples, and pictures were used from the following sources:***

- ***Blitzer Precalculus (Third Edition)***
- ***Onlinemathlearning.com***
- ***Basicmathematics.com***

- ***Tes Tech***
- ***Lumen Learning***
- ***Technology UK***
- ***Michael Van Biezen***
- ***Mathwords***